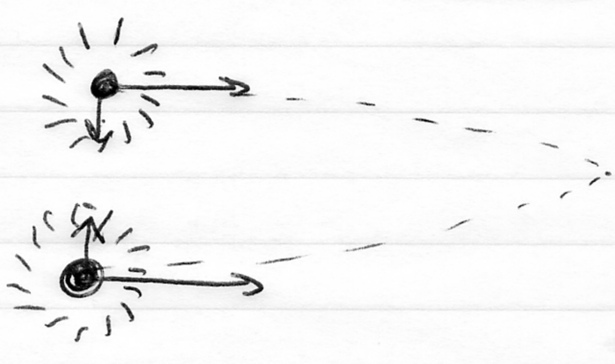
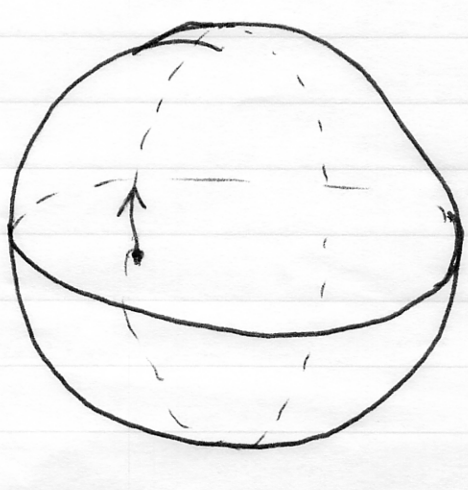
Lecture 1 - The Nature and Uses of Differential Geometry

Imagine two objects floating through space, on parallel ~~courses~~ velocity vectors

Over time, they start to converge... there must be a force pulling together!

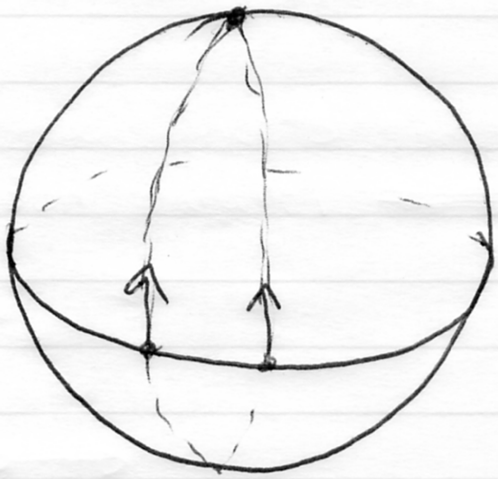


Here's another way to look at it:



One object constrained to a sphere will travel along a great circle (by symmetry).

So two guys will meet at the pole!

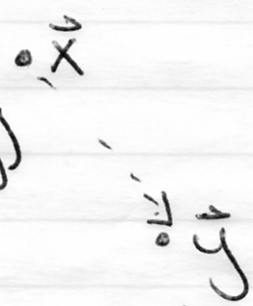


not near light speed

This is the heart of special relativity: the force of gravity can be modelled by the behavior of objects in <u>curved spaces</u>.

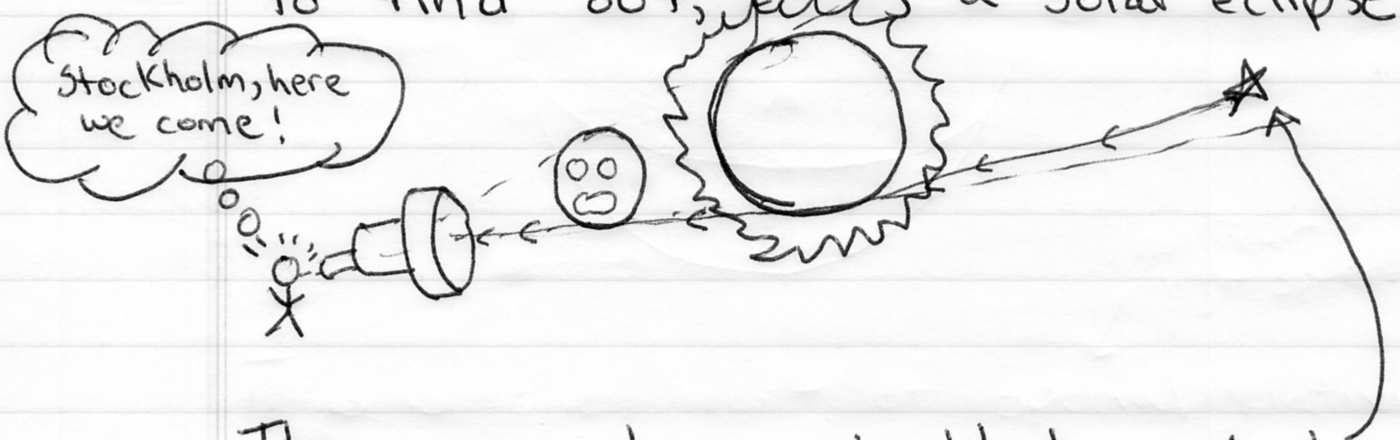
Why bother? In Newtonian physics, the <u>force</u> of gravity is given by the equation

$$F = \frac{\vec{y}-\vec{x}}{|\vec{y}-\vec{x}|^3} \, \text{Mass}(x)\,\text{Mass}(y)$$



This is quite accurate for macroscopic masses. The problem is light.

By assumption, a photon has <u>no mass</u>. So are photons exempt from gravity? A famous experiment took place in South Africa in 1913 to find out, during a solar eclipse:



The moon and sun should have just hidden a certain star from the observers... but instead the mass of the sun had deflected the photons coming from the star "around the corner" and into the telescope!

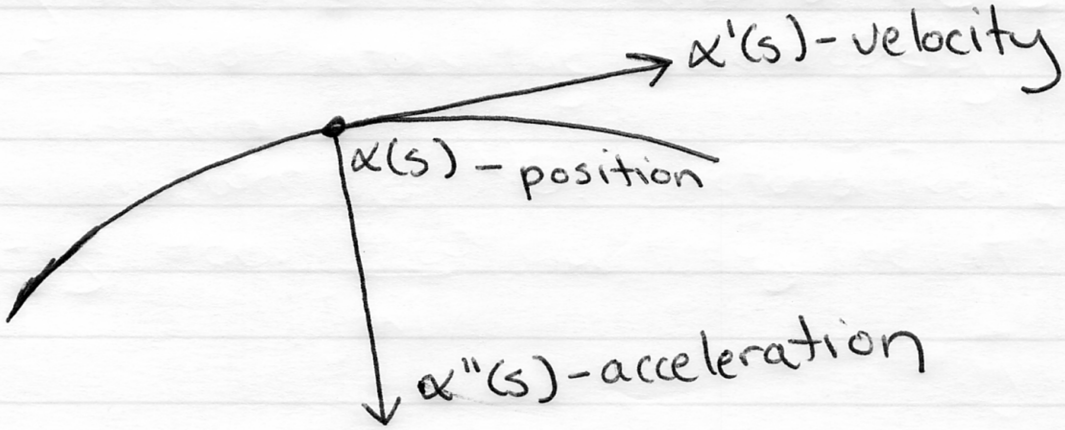
So the answer is NO! Photons feel gravity, too. <u>How</u> do they feel it?

"Definition" A <u>geodesic</u> in a curved space is a (locally) shortest path between two endpoints.

~~The~~ In Euclidean space, geodesics are straight lines. On the sphere, they are great circles. On other spaces... we'll learn how to describe them.

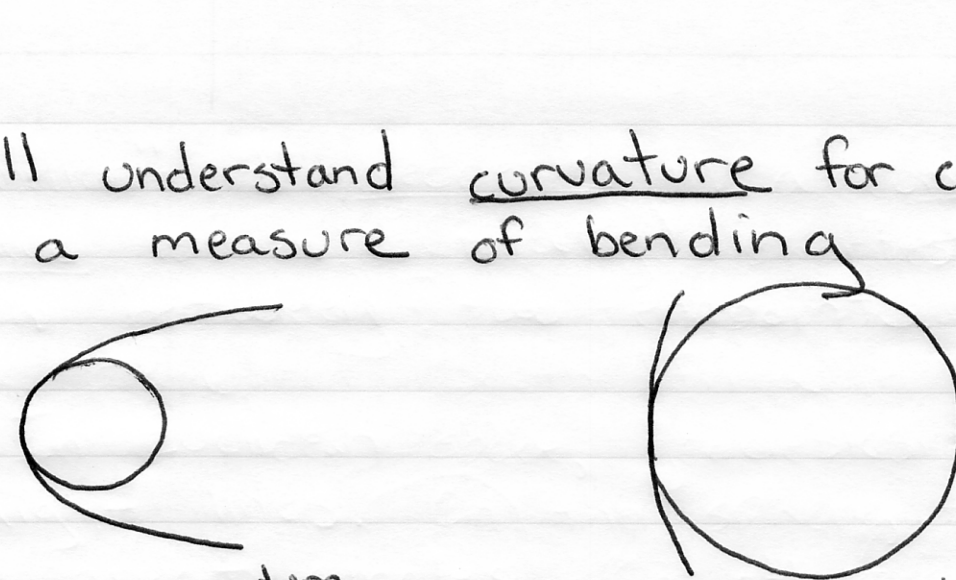
To do so, we'll start by learning about the geometry of <u>parametrized curves</u>:

Definition. A parametrized curve $\alpha(s)$ is a function from $\mathbb{R}$ to $\mathbb{R}^3$.



The local theory of curves is based on understanding their derivatives.

We'll understand <u>curvature</u> for curves as a measure of bending



large curvature
small radius of
tangent circle

small curvature
big circle

And learn remarkable global facts about curvature: If $\kappa(s)$ is the curvature of $\alpha(s)$, we call a <u>critical</u> point of curvature a <u>vertex</u>.

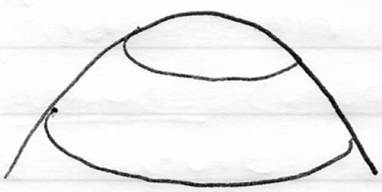
Theorem: Any closed plane convex curve has at least <u>four</u> vertices.

Theorem. Any closed convex plane curve has <u>total</u> <u>curvature</u> $\int \kappa(s)\, ds = 2\pi$.

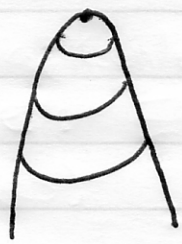
We'll then turn to <u>surfaces</u>.

Definition. A <u>surface</u> is a set in $\mathbb{R}^3$ which can be written locally as the image of a map $f: \mathbb{R}^2 \to \mathbb{R}^3$ at every point.

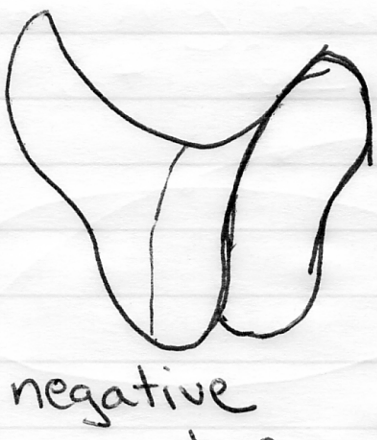
Surfaces will have their own definition of curvature



positive curvature



more positive curvature



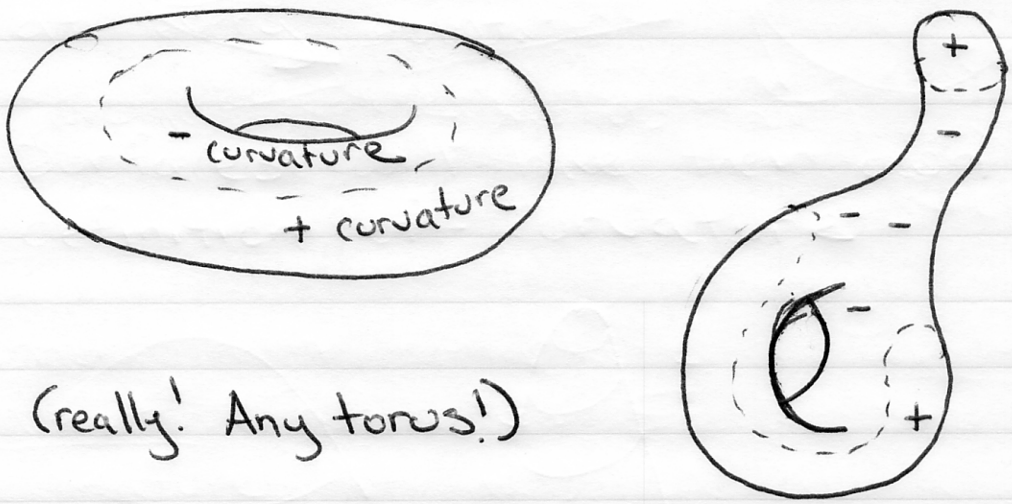
negative curvature

more negative curvature

measure radius of sun 0.5K too much!

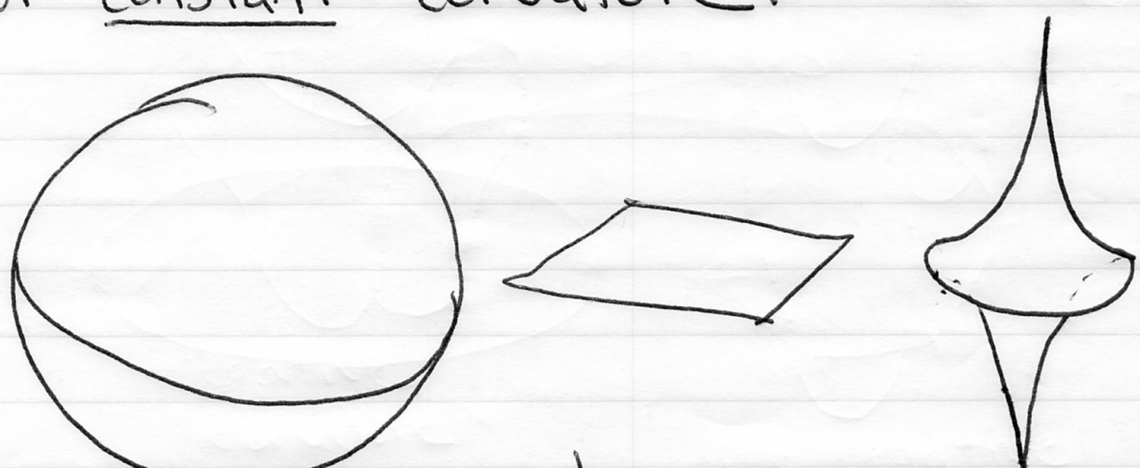
And their own global theorems:

Gauss-Bonnet Theorem. The total curvature of <u>any</u> torus is 0.



(really! Any torus!)

Finally, we'll learn about surfaces of <u>constant</u> curvature:



sphere
$K(u,v) \equiv +1$

plane
$K(u,v) \equiv 0$

pseudosphere
$K(u,v) \equiv -1$

inside out

Structure of the class:

5 homework problems every two weeks, chosen from a list of eight problems.

Every homework problem set labelled; 4 regular problems, 4 challenge problems.

Undergrads	Grads
Must average 1 challenge per homework.	Must average 3 challenge per homework.

There will be two take-home exams during the semester and an in-class final exam. Grad students will have a take-home final.

Points will be broken down (roughly) into:

60% exam grades (20-20-20)
40% homework grade

Office hours are M, T 11-12ish, also can usually catch me Thursday after class.

Email: jason@math.uga.edu

And now... to work!

Definition. We say in this class a function is <u>differentiable</u> if it has derivatives of all orders everywhere it is defined. Other people say <u>smooth</u>.

~~We now~~

Definition. A <u>parametrized differentiable curve</u> is a differentiable map

$$\alpha : (a,b) \longrightarrow \mathbb{R}^3,$$

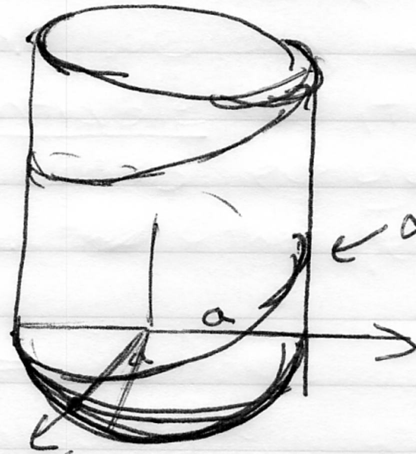
usually written $\alpha(t) = (x(t), y(t), z(t))$.

The vector

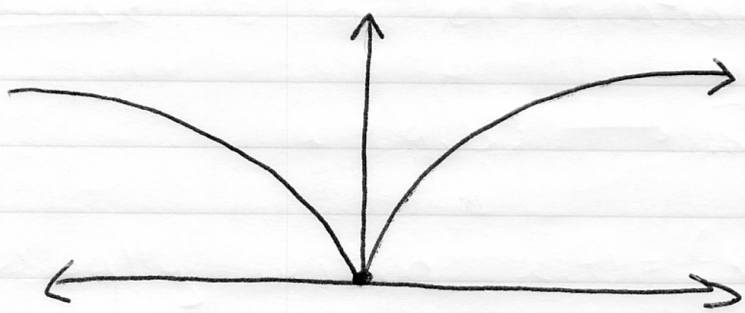
$$\alpha'(t) = (x'(t), y'(t), z'(t))$$

is called the <u>velocity vector</u> or <u>tangent vector</u> to α at t.

Examples.



$\alpha(t) = (a\cos t, a\sin t, bt)$



$\alpha(t) = (t^3, t^2)$

Notice that you don't want to confuse the <u>curve</u> $\alpha(t)$ with its image: the <u>trace</u> of $\alpha(t)$.

(This is an irritant in the theory)

We usually dislike curves like our second example; so we make.

Definition. A parametrized differentiable curve α is <u>regular</u> if $\alpha'(t) \neq \vec{0}$.

~~We define the leng~~

Regular curves are nice; in particular a regular curve cannot have corners like our second example.

We can get even <u>nicer</u> curves:

The <u>arclength</u> of a curve $\alpha(t)$ is defined by

between t_0 and t

$$s(t) = \int_{t_0}^{t} |\alpha'(x)|\,dx.$$

why no t? We used it already!

If $|\alpha'(x)| \equiv 1$, then we observe $s(t) = t - t_0$ and we say α is <u>parametrized</u> <u>by</u> <u>arclength</u>.

We will usually assume curves are parametrized by arclength, in which case they are written $\alpha(s)$.

(This is convenient for theory, but inconvenient for computations; we'll have to use Gray for that stuff.)

Now we dive into curve geometry!

If $\alpha'(s)$ is the tangent line to α, $\alpha''(s)$ tells us how that line is changing; how fast α is pulling away from that line.

Thus

Definition: Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized ~~curve~~ by arclength.

$$|\alpha''(s)| \text{ is called the } \underline{\text{curvature}} \text{ of } \alpha(s)$$

It is usually denoted $\kappa(s)$.