Ropelength and the Geometry of Knots

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Motivating questions

Question

What are the shapes of tightly knotted tubes?

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How do those shapes depend on the topology of the knot?

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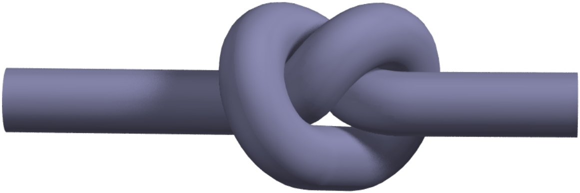
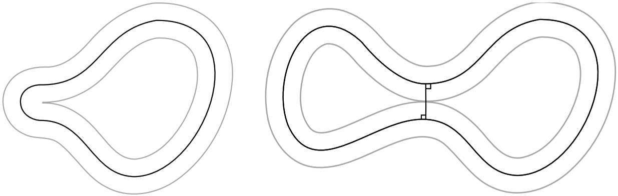


(From numerical computation, true shape unknown.)

Review from Elizabeth Denne's talk:

Definition (Federer 1959)

The *reach* of a space curve is the largest ϵ so that any point in an ϵ-neighborhood of the curve has a unique nearest neighbor on the curve.



Idea

reach(K) *(also called thickness) is controlled by curvature maxima (kinks) and self-distance minima (struts).*

Ropelength

Definition

The ropelength of K is given by $\mathrm{Rop}(K) = \mathrm{Len}(K)/\,\mathrm{reach}(K)$.

Theorem (with Kusner, Sullivan 2002, Gonzalez, De la Llave 2003, Gonzalez, Maddocks, Schuricht, Von der Mosel 2002)

Ropelength minimizers (called tight knots) exist in each knot and link type and are $C^{1,1}$.

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What is the smoothness of a tight knot? Current examples suggest that such a knot is piecewise smooth but not C^2.

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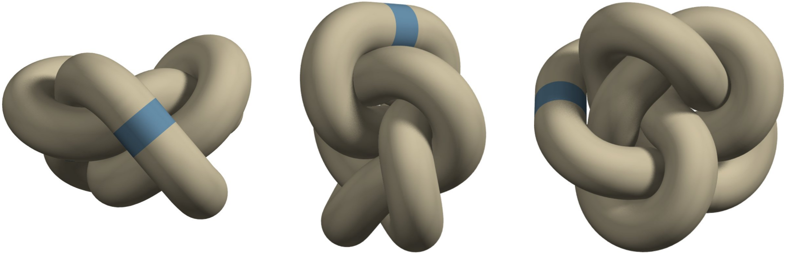
Examples: Why only piecewise smooth?



Theorem (with Fu, Kusner, Sullivan, Wrinkle 2009, cf. Gonzalez, Maddocks 2000, Schuricht, Von der Mosel 2003)

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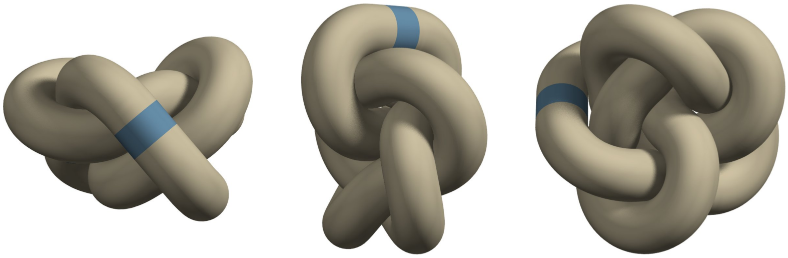

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Lower Bounds on Ropelength

Theorem (Diao 2006)

$$\operatorname{Rop}(K) \geq \frac{1}{2}\left(17.334 + \sqrt{17.334^2 + 64\pi \operatorname{Cr}(K)}\right).$$

Corollary

The unknot has the lowest ropelength of all knots. For any N, there are only a finite number of knots with ropelength $< N$.

Proof.

The ropelength of a tight unknot is $4\pi = 12.566$, less than any knot of higher crossing number. All knots with Rop $< N$ have

$$\operatorname{Cr} < \frac{0.000125}{\pi} N(500N - 8667).$$

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More consequences of this Cr / Rop bound.

Corollary

Hopf link (Rop $= 25.1327$) is the tightest nontrivial link.

Proof.

Evaluating the formula in a few cases,

$\mathrm{Cr}(K)$	3	4	5	$\ldots$	10	11
$\mathrm{Rop}(K) \geq$	23.698	25.286	26.735	$\ldots$	32.704	33.73

So only $\mathrm{Rop}(3_1)$ could be lower than $\mathrm{Rop}(2_1)$. But DDS show $\mathrm{Rop}(3_1) \geq 31.32 > 25.137$. $\square$

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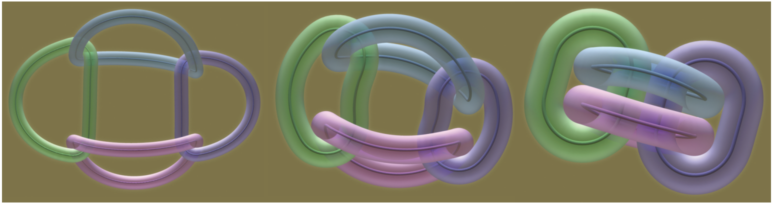
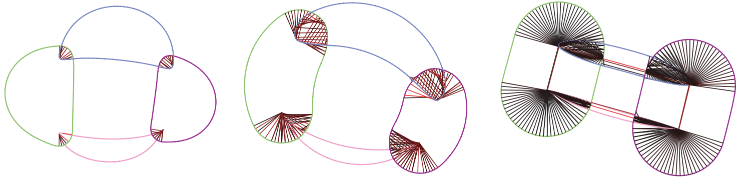
Are these bounds close to tight?

To know, we need approximate numerical data. How to get it?
Simulate the gradient flow of length



. . . with struts entered as new constraints as they form . . .



. . . eventually all motion is stopped by constraints.

Trefoil tightening movie

Show tightening movie 1.

What does this tell you about the math?

Theorem (Rawdon 2000)

Suppose that P is a polygonal knot. Then there exists a $C^{1,1}$ knot K inscribed in P so that $\operatorname{Rop}(P) \geq \operatorname{Rop}(K)$.

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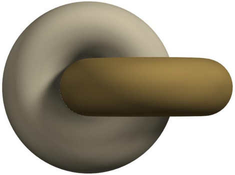

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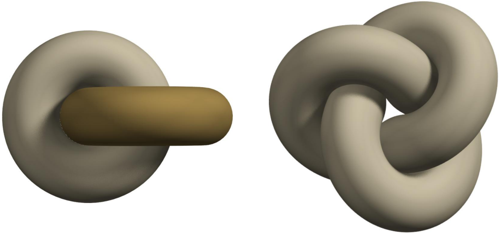

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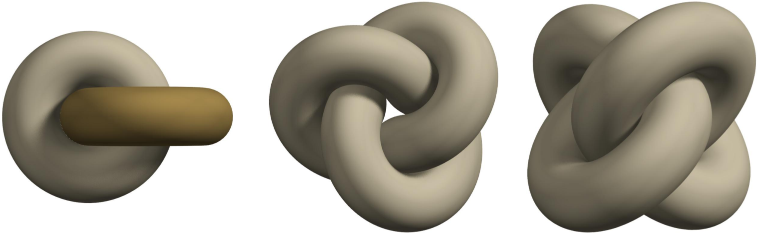

Some more tight polygonal knots . . .

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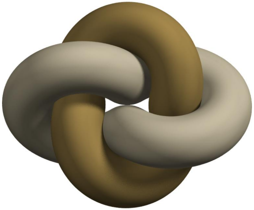

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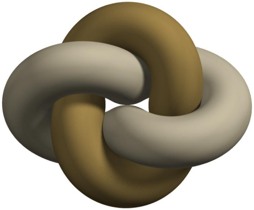

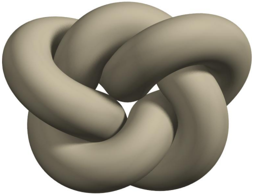

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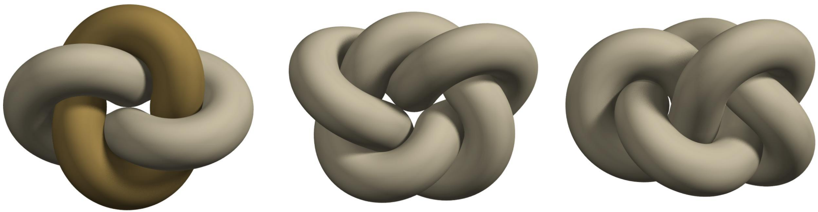

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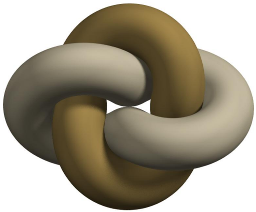
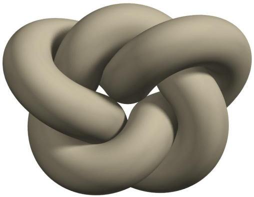
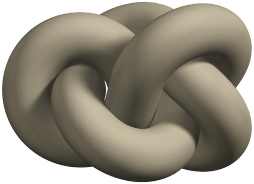
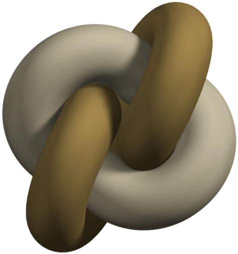

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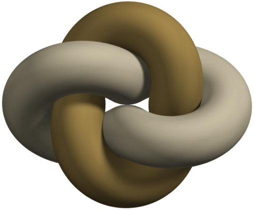

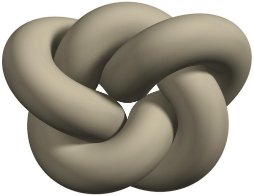

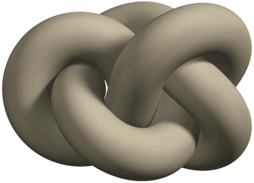

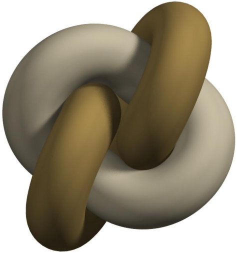

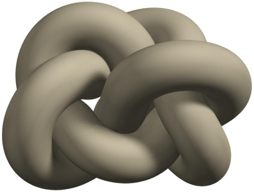

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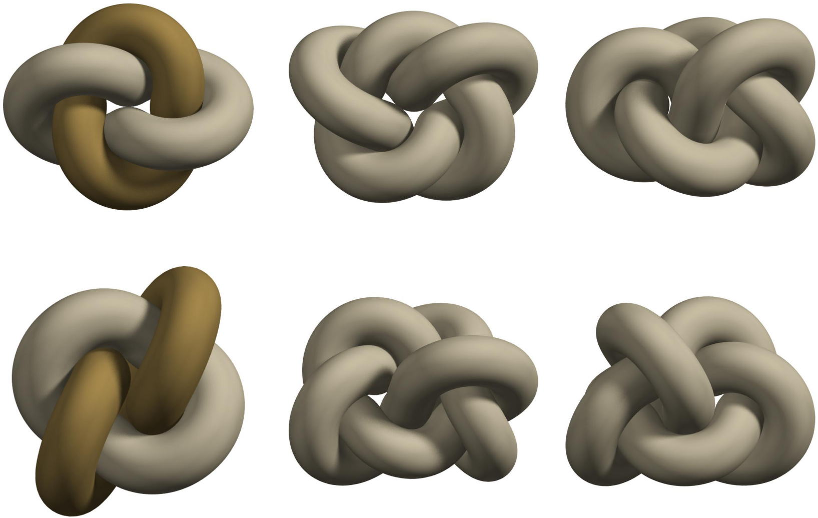
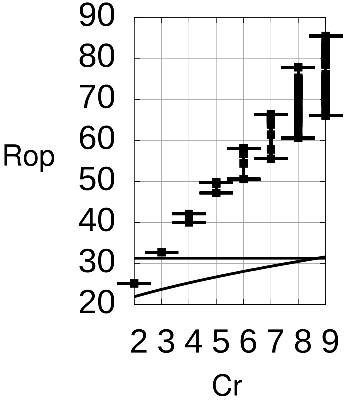


EE (Everybody Else ≤ 10 *crossings*)

Ropelength and Crossing Number vs Data

Question

Find effective Rop *bounds for simple* (<10 *crossing*) *knots.*



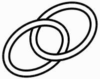
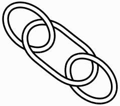
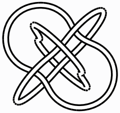
Cr	Rop	Links
3	32.74	3_1
4	$[40.01, 42.09]$	4^2_1, 4_1
5	$[47.20, 49.77]$	5_1, 5^2_1
6	$[50.57, 58.1]$	6^3_3, 6^3_2
7	$[55.53, 66.33]$	7^2_7, 7^2_6
8	$[60.58, 77.83]$	8^3_7, 8^3_4
9	$[66.06, 85.47]$	9^2_{49}, 9^4_1

Wait a minute: how accurate was that data?

Answer

Unknown. We can only check known answers.

Computation (with Ashton, Piatek, Rawdon, `ridgerunner`)

			
Link name	*Hopf link* (2^2_1)	$2^2_1\#2^2_1$	6^3_2
Vertices	216	384	630
Rop *bound*	*25.1389*	*41.7086588*	*58.0146*
Rop	8π	$12\pi+4$	58.0060
Error	0.02%	0.02%	0.01%

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	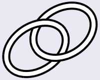	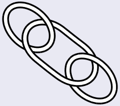	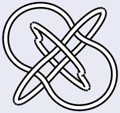
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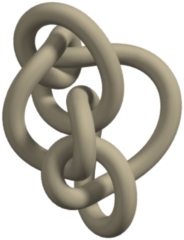
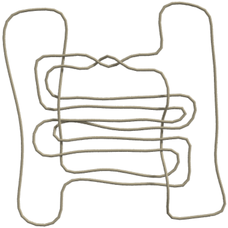
Natural question: Are there local minimum knots?

If we start a knot from two different positions, could we get different “tight” configurations?



10_{137}
KnotPlot (Rob Scharein)



10_{137} (really)
Kawauchi (Ellie Dannenberg)

Movies of the 10_{137} tightenings

We just feed them to the computer and watch it go ...

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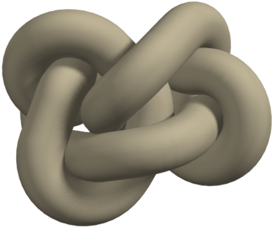
Here are the resulting stopping points for the algorithm side-by-side:



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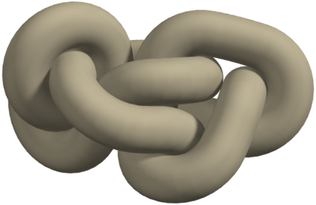

(still) 10_{137}
Kawauchi (Ellie Dannenberg)

Local minimum knots: What can we actually *prove*?

Question

Suppose we have an unknotted open interval of rope and we pull it tight. Is there any tight shape other than the straight line?

Theorem (with Fu, Kusner, Sullivan, Wrinkle)

There are open, ropelength critical, curves with no tube contacts different from the straight line.

Actually, this is a semi-classical question:

Question (Markov-Dubins-Reed-Shepp Car (Sussmann 1995))

What are the length-critical curves with curvature bounded above? How do they depend on boundary conditions?

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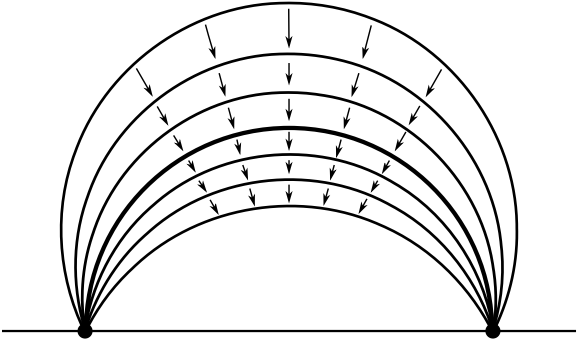
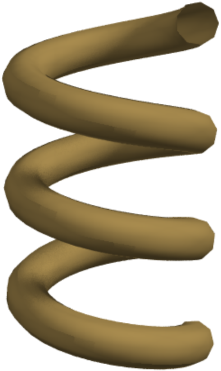
Intuition: What do we expect to be true?

Switch to document camera demo . . .

Option 1: *Some* circle arcs. . .



Option 2: *Certain* helices . . .



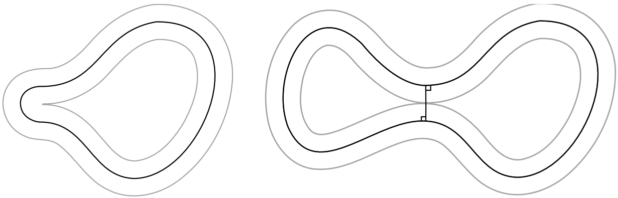
Option 3: Something else . . .

(Numerical solution to known ODE given later in talk.)

Just to remember . . .

Definition

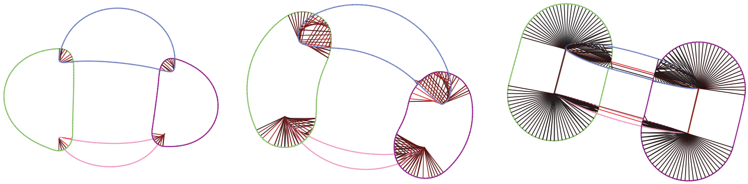
The *reach* of a space curve is the largest ϵ so that any point in an ϵ-neighborhood of the curve has a unique nearest neighbor on the curve.



Idea

Reach is controlled by curvature (kinks) and pairs of closest approach (struts).

Strut measures



Definition

A *strut measure* is a non-negative Radon measure on the struts representing a compression force pointing outwards.

Definition

A strut force measure $\mathbb{S}$ on L is the vector-valued Radon measure defined at each point p of L by integrating a strut measure over all the struts with an endpoint at p.

Main Theorem

We can find an Euler-Lagrange equation for ropelength-critical curves. Think of the strut force measure as an (infinite) set of Lagrange multipliers. Then

Idea of theorem (CFKSW (2009))

Suppose L is thickness 1, and the parts of L with maximum curvature are nice. Then at each point on the curve

$$\begin{aligned}\textit{total force from self contacts} &= \textit{elastic force}\\ &\quad + \textit{force transmitted through kinks.}\end{aligned}$$

Main Theorem

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Theorem (CFKSW (2009))

Suppose L is λ-critical, and that Kink *is included in a subarc on which L is regulated. Then $\exists$ a strut force measure* $\mathbb{S}$ *and a nonnegative lower semicontinuous function $\phi \in \mathrm{BV}(L)$ such that $(\phi N)' \in \mathrm{BV}(L)$, with*

$$\mathbb{S}|_{\text{interior } L} = -\left((1-2\phi)T - \frac{\lambda}{2}(\phi N)'\right)'\Bigg|_{\text{interior } L}.$$

If p is a fixed endpoint of L, $\phi(p) = 0$.

Main Theorem

We can find an Euler-Lagrange equation for ropelength-critical curves. Think of the strut force measure as an (infinite) set of Lagrange multipliers. Then

Theorem (CFKSW (2009))

Suppose L is λ-critical and piecewise C^2 and L has no struts. Then L is piecewise analytic and there exists a nonnegative lower semicontinuous function $\phi \in \mathrm{BV}(L)$ such that $(\phi N)' \in \mathrm{BV}(L)$ and

$$(1-2\phi)T - \frac{\lambda}{2}(\phi N)' \equiv V_0 = \textit{constant}.$$

We can find an Euler-Lagrange equation for ropelength-critical curves. Think of the strut force measure as an (infinite) set of Lagrange multipliers. Then

Theorem (CFKSW (2009))

Suppose L is λ-critical and piecewise C^2 and L has no struts. Then L is piecewise analytic and there exists a nonnegative smooth function ϕ so that

$$\phi'' + (\kappa^2 - \tau^2)\phi = \kappa^2 \quad (1)$$

$$\tau\phi^2 = c \quad (2)$$

for some constant c. Since $\kappa = 2/\lambda =$ constant, this is a system of ODE for τ and ϕ with initial conditions specified by c and $\phi(0)$, and a constant solution $\phi = \phi_0(c)$.

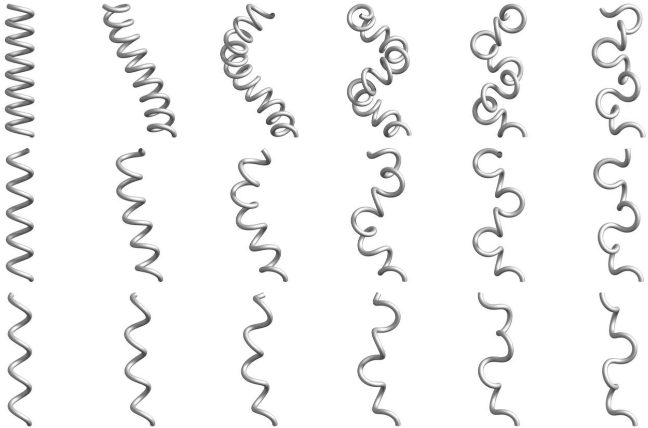
Pictures of solutions

$\phi_0(c)\times$ 1   1.125   1.25   1.5   1.75   2

$c = \frac{1}{2}$

$c = 1$

$c = 3$



The case $c = 0$.

Suppose first that $c = 0$. Either $\phi = 0$ (the curve is a line) or $\tau = 0$ (the curve is planar, and hence a circle arc).

Proposition

A circular arc of radius $\frac{\lambda}{2}$ with fixed endpoints is critical with no strut force measure $\iff$ its angle exceeds π.

Proof.

Rewriting (1) in terms of an angle θ along an arc $\theta \in [0, \theta_0]$, we get the system

$$\phi'' + \phi = 1, \quad \phi(0) = \phi(\theta_0) = 0. \tag{3}$$

This has a positive solution ($\phi = 1 - \cos\theta + B\sin\theta$) is ≥ 0 $\iff \theta_0 > \pi$. $\square$

Lemma

A critical curve without strut force measure and with ϕ constant is a helix with $|\tau| < \kappa$. Further, the conserved vector V_0 points in the direction of the axis of the helix.

Proof.

Recall that $V_0 = (1 - 2\phi)T - \frac{\lambda}{2}(\phi N)'$ so

$$T \cdot V_0 = (1 - 2\phi) - \frac{\phi}{\kappa} T \cdot N' = 1 - \phi = const. \tag{4}$$

Thus (cf. doCarmo) κ/τ (and so τ) are constant.
Now $\phi'' = 0$ so (1) becomes $(\kappa^2 - \tau^2)\phi = \kappa^2$.
Since $\phi \geq 0$, we have $\kappa^2 > \tau^2$. □

The general case.

We may assume $c \neq 0$, so ϕ is not always zero. Where $\phi > 0$, we have $\tau = c/\phi^2$, so (1) and (2) become the semilinear ODE

$$\phi'' = \kappa^2(1-\phi) + \frac{c}{\phi^3} := f_c(\phi). \tag{5}$$

Lemma

All solutions of (5) *are positive periodic functions.*

Proof.

(5) is an autonomous system with integrating function

$$F(x,y) = \left(\frac{\kappa^2}{2}x^2 + \frac{1}{2}y^2\right) - \kappa^2 x + \frac{c^2}{2x^2} = \text{const}, \tag{6}$$

where $x = \phi$ and $y = \phi'$. $\square$

Can these curves ever close?

Theorem (CFKSW (2009))

Any closed piecewise C^2 λ-critical curve with no strut force measure is a circle of radius $\lambda/2$.

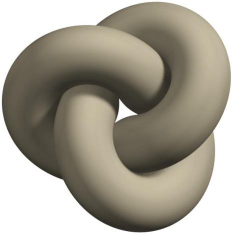
Proof.

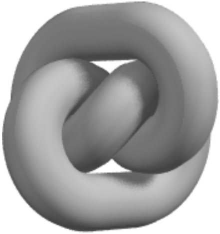
We have reduced to the case $\phi > 0$ with period P. Recall $T \cdot V_0 = 1 - \phi$. Solving (5) for $1 - \phi$, we see $1 - \phi = \frac{1}{\kappa^2}\phi'' - \frac{c}{\kappa^2\phi^3}$. So we have

$$\int_0^P T \cdot V_0 \, \mathrm{d}s = \int_0^P \frac{1}{\kappa^2}\phi'' - \frac{c}{\kappa^2\phi^3} \, \mathrm{d}s = -\frac{c}{\kappa^2}\int_0^P \phi^{-3} \, \mathrm{d}s. \qquad (7)$$

This $\neq 0$, since $c \neq 0$ and $\phi > 0$. So over each period the curve moves a constant distance in the V_0 direction. $\square$

Alternate critical configurations for (some) knots

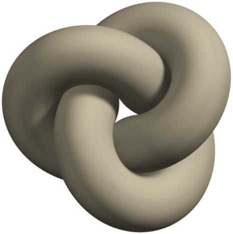


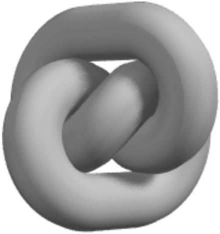


Remark

It would be really nice to extend this strategy to find an "alternate" or Gordian unknot, but you can't, because the round circle already has a symmetry of every period.

Alternate critical configurations for (some) knots





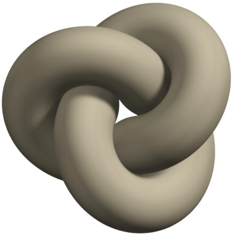
Theorem (with Fu, Kusner, Sullivan, Wrinkle, in preparation)

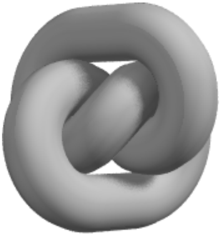
There is another critical configuration of 3_1 with 2-fold symmetry.

Proof.

The proof is based on a symmetric version of the criticality theorem. There should be a critical configuration with 3-fold and with 2-fold symmetry (there is no configuration with both symmetries.) □

Alternate critical configurations for (some) knots





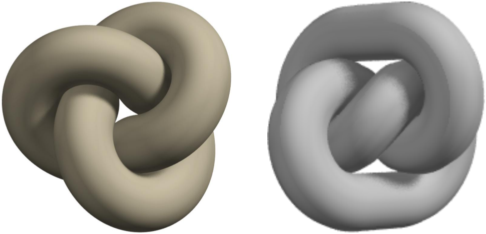
Theorem (with Fu, Kusner, Sullivan, Wrinkle, in preparation)

There is another critical configuration of 3_1 *with* 2*-fold symmetry.*

Proof.

The proof is based on a symmetric version of the criticality theorem. There should be a critical configuration with 3-fold and with 2-fold symmetry (there is no configuration with both symmetries.) □

Alternate critical configurations for (some) knots

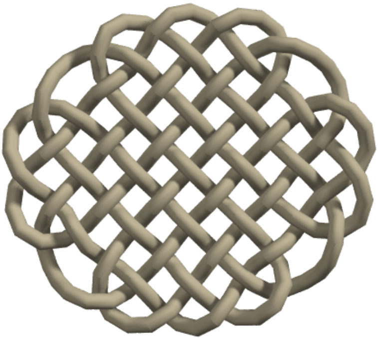


Remark

It would be really nice to extend this strategy to find an "alternate" or Gordian unknot, but you can't, because the round circle already has a symmetry of every period.

Open questions: What about large-scale tight knots?

With our computer algorithm in place, it looks like a good idea to check out large knots.



Ashley knot (ie. your favorite sweater)
KnotPlot

Movies (of putting your favorite sweater in the dryer)

This is why we have simulation: some experiments are just too cruel to perform in real life. (No sweaters were harmed in the making of this film.)

Observation

Notice that the sweater became (sadly) nonplanar after being tightened, but did not completely collapse. It seems like it takes a knot like this to form a large blob.

blob (KnotPlot)

Movies of the blob

(No blobs were harmed in the making of this film either.)

Thank you!

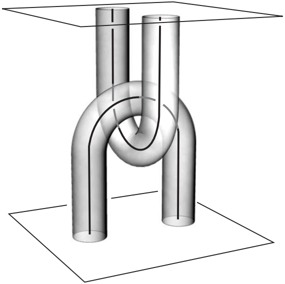
Thank you for inviting me!

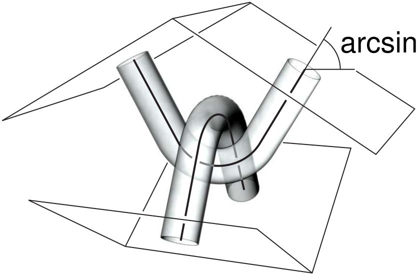
Thank you!

Thank you for inviting me!
And be sure to see Ellie's talk tomorrow . . .

Another solution: Clasps

What happens when a rope is pulled over another?





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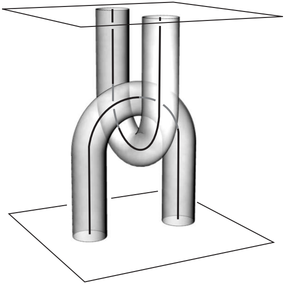
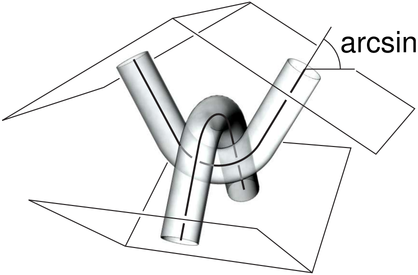
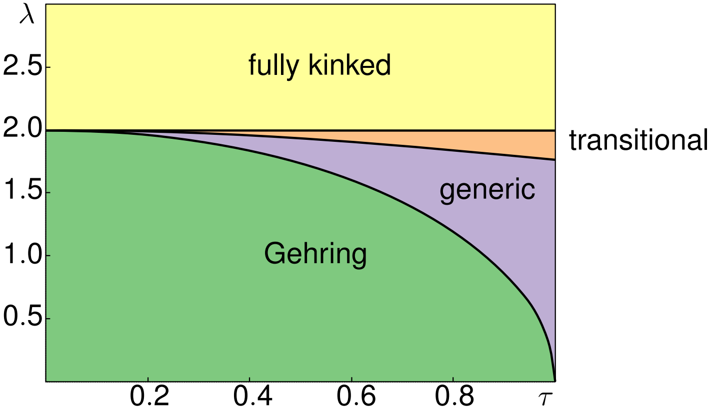


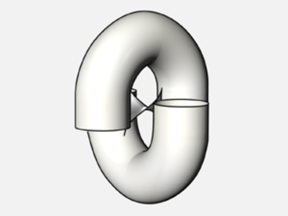


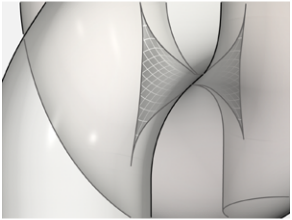
It depends on the angle (τ) and the stiffness (λ) of the rope.

Four types of clasps



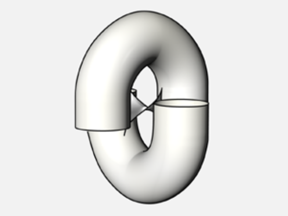
Gehring clasp (CFSKW 2006)

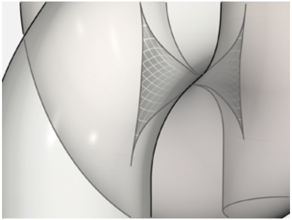




- δ length balanced against strut force only.
- Curvature given explicitly, position as an elliptic integral.
- Small gap between the two tubes.
- Curvature unbounded at tip.

Gehring clasp (CFSKW 2006)





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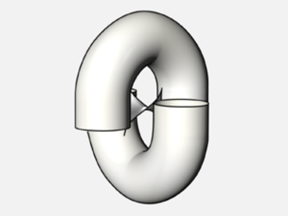

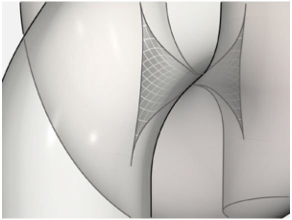

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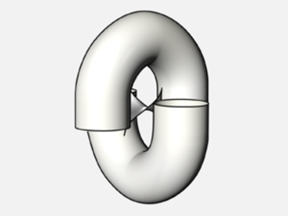

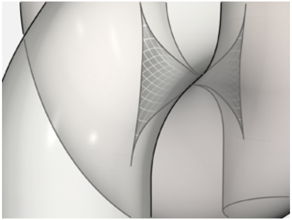

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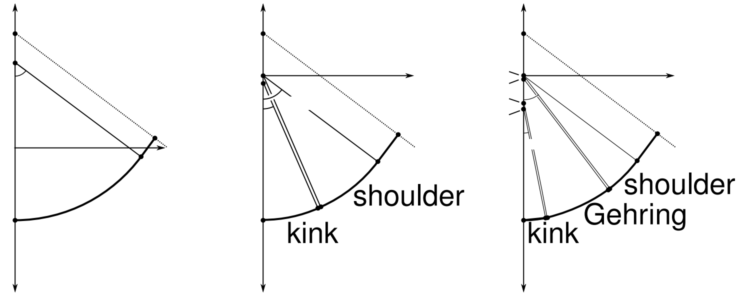
Kinked, Transitional, Generic Clasps



Kinked Clasp Transitional Clasp Generic Clasp