REGULAR POLYTOPES

SECOND EDITION

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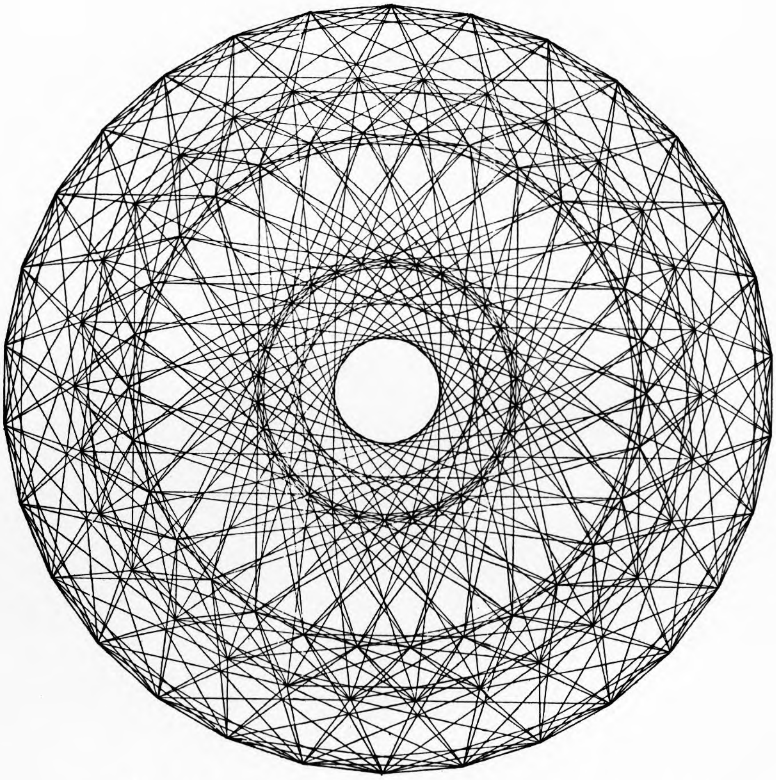
With eight Plates and numerous Diagrams

THE MACMILLAN COMPANY, NEW YORK
COLLIER-MACMILLAN LIMITED, LONDON



The four-dimensional polytope $\{3, 3, 5\}$, drawn by van Oss
(cf. Fig. 13·6B on page 250).

CHAPTER III

ROTATION GROUPS

THIS chapter provides an introduction to the theory of groups, illustrated by the symmetry groups of the Platonic solids. We shall find coordinates for the vertices of these solids, and examine the cases where one can be inscribed in another. Finally, we shall see that every finite group of displacements is the group of rotational symmetry operations of a regular polygon or polyhedron.

3·1. Congruent transformations. Two figures are said to be *congruent* if the distances between corresponding pairs of points are equal, in which case the angles between corresponding pairs of lines are likewise equal. In particular, two trihedra (or trihedral solid angles) are congruent if the three face-angles of one are equal to respective face-angles of the other. Two such trihedra are said to be *directly* congruent (or " superposable ") if they have the same sense (right- or left-handed), but *enantiomorphous* if they have opposite senses. The same distinction can be applied to figures of any kind, by the following device.

Any point **P** is located with reference to a given trihedron by its (oblique) Cartesian coordinates x, y, z. Let **P**′ be the point whose coordinates, referred to a congruent trihedron, are the same x, y, z. If we suppose the two trihedra to be fixed, every **P** determines a unique **P**′, and vice versa. This correspondence is called a *congruent transformation*, **P**′ being the transform of **P**. If another point **Q** is transformed into **Q**′, we have a definite formula for the distance **PQ** in terms of the coordinates, which shows that **P**′**Q**′=**PQ**. In other words, a congruent transformation is a point-to-point correspondence preserving distance. It is said to be *direct* or *opposite* according as the two trihedra are directly congruent or enantiomorphous, i.e., according as the transformation preserves or reverses sense. Hence the product (resultant) of two direct or two opposite transformations is direct, whereas the product of a direct transformation and an opposite transformation (in either order) is opposite. (In fact, the composition of direct and opposite transformations resembles the multiplication of positive and negative numbers, or the addition of even and odd numbers.) A direct transformation is often called a *displacement*, as it can be achieved by a rigid motion. Any two congruent figures are related by a congruent transformation, direct or opposite. Two identical left shoes are directly congruent; a pair of shoes are enantiomorphous. (Some authors use the words "congruent" and "symmetric" where we use "directly congruent" and "enantiomorphous".)

We shall find that all congruent transformations can be derived from three "primitive" transformations: *translation* (in a certain direction, through a certain distance), *rotation* (about a certain line or *axis*, through a certain angle), and *reflection* (in a certain plane). Evidently the first two are direct, while the third is opposite.

There is an analogous theory in space of any number of dimensions. In two dimensions we rotate about a point, reflect in a line, and a congruent transformation is defined in terms of two congruent *angles*. In one dimension we reflect in a point, and

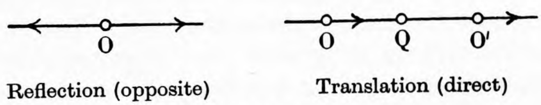


FIG. 3·1A

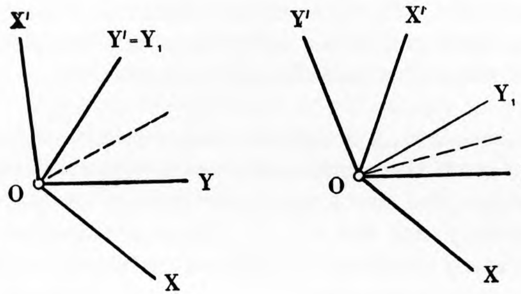
a congruent transformation is defined in terms of two *rays* (or "half lines"). In this simplest case, if any point $\mathbf{O}$ is left invariant, the transformation is the reflection in $\mathbf{O}$, unless it is merely the *identity* (which leaves *every* point invariant); but if there is no invariant point, it is a translation, i.e., the product of reflections in two points ($\mathbf{O}$ and $\mathbf{Q}$, in Fig. 3·1A).

In two dimensions, a congruent transformation that leaves a point $\mathbf{O}$ invariant is either a reflection or a rotation (according as it is opposite or direct). For, the transformation from an angle $\mathbf{XOY}$ to a congruent angle $\mathbf{X'OY'}$ (Fig. 3·1B) can be achieved as follows. By reflection in the bisector of $\angle\mathbf{XOX'}$, $\angle\mathbf{XOY}$ is transformed into $\angle\mathbf{X'OY_1}$. Since this is congruent to $\angle\mathbf{X'OY'}$, the ray $\mathbf{OY'}$ either coincides with $\mathbf{OY_1}$ or is its image by reflection in $\mathbf{OX'}$. In the former case the one reflection suffices; in the latter, it has to be combined with the reflection in $\mathbf{OX'}$, and the product is the rotation through $\angle\mathbf{XOX'}$ (which is twice the angle between the two reflecting lines).

In particular, the product of reflections in two perpendicular lines is a rotation through π or *half-turn*. In this single case, it is immaterial which reflection is performed first; in other words, two reflections *commute* if their lines are perpendicular. It is important to notice that the half-turn about **O** is the product of reflections in *any* two perpendicular lines through **O**.

A plane congruent transformation without any invariant point is the product of two or three reflections (according as it is direct or opposite). For, in transforming an angle **XOY** into a congruent angle **X′O′Y′**, we can begin by reflecting in the perpendicular bisector of **OO′**, and then use one or two further reflections, as above.

The product of two reflections is a translation or a rotation,



Reflection (opposite) Rotation (direct)

FIG. 3·1B

according as the reflecting lines are parallel or intersect. Hence *every plane displacement is either a translation or a rotation.**

In the product of three reflections, we can always arrange that one of the reflecting lines shall be perpendicular to both the others. The following is perhaps not the simplest proof, but it is one that generalizes easily to any number of dimensions. If we regard a congruent transformation as operating on pencils of parallel rays (instead of operating on points), we can say that a translation has no effect: it leaves every pencil invariant. Since each pencil can be represented by that one of its rays which passes through a fixed point **O**, any congruent transformation gives rise to an "induced" congruent transformation operating on the rays that emanate from **O**: congruent because of the preservation of angles.

If the given transformation is opposite, so is the induced

* Kelvin and Tait 1, p. 60.

transformation. But the latter, leaving **O** invariant, can only be a reflection, say the reflection in **OQ**. This leaves **O** and **Q** invariant; therefore the given transformation leaves the *direction* **OQ** invariant. Consider the product of the given transformation with the reflection in any line, **p**, perpendicular to **OQ**. This is a direct transformation which reverses the direction **OQ**; i.e., it is a half-turn. Hence the given transformation is the product of a half-turn with the reflection in **p**. But the half-turn is the product of reflections in two perpendicular lines, which may be chosen perpendicular and parallel to **p**. Thus we have altogether three reflections, of which the last two can be combined to form a translation. The general opposite transformation is now reduced to the product of a reflection and a translation which commute, the reflecting line being in the direction of the translation. This kind of transformation is called a *glide-reflection*.

In three dimensions, a congruent transformation that leaves a point **O** invariant is the product of at most three reflections: one to bring together the two x-axes, another for the y-axes, and a third (if necessary) for the z-axes. Since one further reflection will suffice to bring together two different origins (i.e., the vertices of the two congruent trihedra),

3·11. *Every congruent transformation is the product of at most four reflections.*

Since the product of two opposite transformations is direct, a product of reflections is direct or opposite according as the number of reflections is even or odd. Hence every direct transformation is the product of two or four reflections, and every opposite transformation is either a single reflection or a product of three.

The product of reflections in two parallel planes is a translation in the perpendicular direction through twice the distance between the planes, and the product of reflections in two intersecting planes is a rotation about the line of intersection through twice the angle between them. Two reflections commute if their planes are perpendicular, in which case their product is a *half-turn* (or "reflection in a line").

Since the product of three reflections is opposite, a direct transformation with an invariant point **O** can only be the product of reflections in *two* planes through **O**, i.e., a rotation. Thus

3·12. *Every displacement leaving one point invariant is a rotation.**

Consequently the product of two rotations with intersecting axes is another rotation.

The three "primitive" transformations (viz., translation, rotation, and reflection), taken in commutative pairs, form the following three products. A *screw-displacement* is a rotation combined with a translation in the axial direction. A *glide-reflection* is a reflection combined with a translation whose direction is that of a line lying in the reflecting plane. A *rotatory-reflection* is a rotation combined with the reflection in a plane perpendicular to the axis. In the last case, if the rotation is a half-turn, the rotatory-reflection is an *inversion* (or "reflection in a point"), and the direction of the axis is indeterminate. In fact, an inversion is the product of reflections in any three perpendicular planes through its centre ; e.g., reflections in the axial planes of a Cartesian frame reverse the signs of x, y, z, respectively, and their product transforms (x, y, z) into $(-x, -y, -z)$.

We proceed to prove that every congruent transformation is of one of the above kinds.

An *opposite* transformation, being the product of (at most) three reflections, leaves invariant either a point or two parallel planes (and all planes parallel to them). The latter possibility is the limiting case of the former when the invariant point recedes to infinity ; it arises when the three reflecting planes are all perpendicular to one plane, instead of forming a trihedron.

If there is an invariant point O, consider the product of the given (opposite) transformation with the inversion in O. This direct transformation, leaving O invariant, must be a rotation. Hence the given transformation is a "rotatory-inversion", the product of a rotation with the inversion in a point on its axis. By regarding the inversion as a special rotatory-reflection,† we see that a rotatory-inversion involving rotation through angle θ is the same as a rotatory-reflection involving rotation through $\theta-\pi$. Hence every opposite transformation leaving one point invariant is a rotatory-reflection.

* Kelvin and Tait **1**, p. 69.

† Crystallographers prefer to take translation, rotation, and *inversion* as "primitive" transformations, and to regard a reflection as a special rotatory-inversion. See Hilton **1**, Donnay **1**.

If, on the other hand, it is two parallel planes that are invariant, the transformation is essentially two-dimensional: what happens in one of the two planes happens also in the other and in all parallel planes. By the two-dimensional theory, we then have a glide-reflection. Hence

3·13. *Every opposite congruent transformation is either a rotatory-reflection or a glide-reflection* (including a pure reflection as a special case).

In order to analyse the general displacement or *direct* transformation, we first regard the transformation as operating on bundles of parallel rays, represented by single rays through a fixed point **O**. The induced transformation, leaving **O** invariant, is still direct, and so can only be a rotation. The direction of the axis, **OQ**, of this rotation, must be invariant for the original displacement as well. Let ϖ be any plane perpendicular to **OQ**. The product of the displacement with the reflection in ϖ is an opposite transformation which reverses the direction **OQ**, i.e., a rotatory-reflection or glide-reflection whose reflecting plane is parallel to ϖ. Reflecting in ϖ again, and remembering that the product of reflections in two parallel planes is a translation, we express the displacement as the product of a rotation or translation with a translation, i.e., as either a "rotatory-translation" or a pure translation. The latter alternative can be disregarded, as being merely a special case of the former. This "rotatory-translation" is the product of a rotation about a line parallel to **OQ** with a translation in the direction **OQ** (or **QO**), i.e., a screw-displacement. Hence

3·14. *Every displacement is a screw-displacement* (including, in particular, a rotation or a translation).*

3·2. Transformations in general. The concept of a congruent transformation, applied to figures in space, can be generalized to that of a one-to-one transformation applied to any set of elements.† When we speak of the resultant of two transformations as their "product", we are making use of the analogy that exists between transformations and numbers. We shall often use letters R, S, . . . to denote transformations, and write RS

* Kelvin and Tait 1, pp. 78-79.
† See Birkhoff and MacLane 1, pp. 124-127.

for the resultant of R and S (in that order). This notation is justified by the validity of the associative law

3·21 $$(\mathrm{RS})\mathrm{T} = \mathrm{R}(\mathrm{ST}).$$

Since a number is unchanged when multiplied by 1, it is natural to use the same symbol 1 for the "identical transformation" or *identity* (which enters our discussion as the translation through no distance, and again as the rotation through angle 0 or through a complete turn). Pushing the analogy farther, we let R^p denote the p-fold application of R; e.g., if R is a rotation through θ, R^p is the rotation through $p\theta$ about the same axis. A transformation R is said to be periodic if there is a positive integer p such that $\mathrm{R}^p=1$; then its *period* is the smallest p for which this happens. We also let R^{-1} denote the *inverse* of R, which neutralizes the effect of R, so that $\mathrm{RR}^{-1}=1=\mathrm{R}^{-1}\mathrm{R}$. If R is of period p, we have $\mathrm{R}^{-1}=\mathrm{R}^{p-1}$. In particular, a transformation of period 2 (such as a reflection, half-turn, or inversion) is its own inverse.

The general formula for the inverse of a product is easily seen to be

$$(\mathrm{RS}\ldots\mathrm{T})^{-1} = \mathrm{T}^{-1}\ldots\mathrm{S}^{-1}\,\mathrm{R}^{-1}.$$

If R, etc. are of period 2, this is the same as $\mathrm{T}\ldots\mathrm{SR}$; e.g., if R and S are reflections in parallel planes, the products RS and SR are two inverse translations, proceeding in opposite directions. The analogy with numbers might be regarded as breaking down in the general failure of the commutative law $\mathrm{SR}=\mathrm{RS}$; but there are generalized numbers, such as quaternions, which likewise need not commute.

Let $\mathbf{x}$ denote any figure to which a transformation is applied. If T transforms $\mathbf{x}$ into $\mathbf{x}'$ (so that T^{-1} transforms $\mathbf{x}'$ into $\mathbf{x}$), we write

$$\mathbf{x}' = \mathbf{x}^{\mathrm{T}}.$$

This notation is justified by the fact that $(\mathbf{x}^{\mathrm{T}})^{\mathrm{S}}=\mathbf{x}^{\mathrm{TS}}$. If S transforms the pair of figures $(\mathbf{x}, \mathbf{x}^{\mathrm{T}})$ into $(\mathbf{x}_1, \mathbf{x}_1^{\mathrm{T}_1})$, we say that S transforms T into T_1, and write

$$\mathrm{T}_1 = \mathrm{T}^{\mathrm{S}}.$$

(We may speak of this as "T transformed by S"; e.g., if T is a rotation about an axis $\mathbf{l}$, then T^{S} is the rotation through the same angle about the transformed axis $\mathbf{l}^{\mathrm{S}}$.) Since $\mathbf{x}_1=\mathbf{x}^{\mathrm{S}}$ and $\mathbf{x}_1^{\mathrm{T}_1}=(\mathbf{x}^{\mathrm{T}})^{\mathrm{S}}$, we have $\mathbf{x}^{\mathrm{ST}_1}=\mathbf{x}^{\mathrm{TS}}$ for every $\mathbf{x}$. Hence $\mathrm{ST}_1=\mathrm{TS}$, and

$$\mathrm{T}^{\mathrm{S}} = \mathrm{S}^{-1}\mathrm{TS}.$$

Transforming a product, we find that

$$(\mathrm{TU})^{\mathrm{S}} = \mathrm{S}^{-1}\mathrm{TUS} = \mathrm{S}^{-1}\mathrm{TSS}^{-1}\mathrm{US} = \mathrm{T}^{\mathrm{S}}\mathrm{U}^{\mathrm{S}}.$$

Hence, for any integer p, $(\mathrm{T}^p)^{\mathrm{S}} = (\mathrm{T}^{\mathrm{S}})^p$.

If S and T commute, so that TS=ST and $\mathrm{T}^{\mathrm{S}}=\mathrm{T}$, we say that T is *invariant* under transformation by S.

The "figure" $\mathbf{x}$ need not be geometrical; e.g., it could be a number or *variable*, in which case $\mathbf{x}^{\mathrm{T}}$ is a function of this variable, and a more customary notation is $\mathrm{T}(\mathbf{x})$. (The particular transformations

$$\mathbf{x}' = \mathbf{x}^t,$$

where t takes various *numerical* values, are seen to combine among themselves just like the numbers t.) Again, $\mathbf{x}$ could be a discrete set of objects in assigned positions, and $\mathbf{x}^{\mathrm{T}}$ the same set rearranged; then T is a *permutation*.

The two alternative notations currently used for permutations are illustrated by the symbols

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} & \mathbf{g} \\ \mathbf{c} & \mathbf{g} & \mathbf{e} & \mathbf{d} & \mathbf{a} & \mathbf{f} & \mathbf{b} \end{pmatrix} \quad \text{and} \quad (\mathbf{a\ c\ e})\ (\mathbf{b\ g})$$

for the permutation of seven letters that replaces **a**, **b**, **c**, **e**, **g**, by **c**, **g**, **e**, **a**, **b**, while leaving **d** and **f** unchanged. In the latter notation, which we shall use exclusively, the two parts **(a c e)** and **(b g)** are called *cycles*. Clearly, every permutation is a product of cycles involving distinct sets of objects. It is sometimes desirable to include all the objects, e.g., to write

$$(\mathbf{a\ c\ e})\ (\mathbf{b\ g})\ (\mathbf{d})\ (\mathbf{f}),$$

calling **(d)** and **(f)** "cycles of period 1". A *transposition* is a single cycle of period 2, such as **(b g)**, which merely interchanges two of the objects.

A permutation is said to be *even* or *odd* according to the parity of the number of cycles of even period; e.g., **(a c e) (b g)** is an odd permutation. When a permutation is multiplied by a transposition, its parity is reversed. For, if $(\mathbf{a}_1\ \mathbf{b}_1)$ is the transposition, $\mathbf{a}_1$ and $\mathbf{b}_1$ must either occur in the same cycle of the given permutation or in two different cycles. Since

$$(\mathbf{a}_1 \ldots \mathbf{a}_r\ \mathbf{b}_1 \ldots \mathbf{b}_s)\ (\mathbf{a}_1\,\mathbf{b}_1) = (\mathbf{a}_1 \ldots \mathbf{a}_r)\ (\mathbf{b}_1 \ldots \mathbf{b}_s)$$

and $$(\mathbf{a}_1 \ldots \mathbf{a}_r)\ (\mathbf{b}_1 \ldots \mathbf{b}_s)\ (\mathbf{a}_1\,\mathbf{b}_1) = (\mathbf{a}_1 \ldots \mathbf{a}_r\ \mathbf{b}_1 \ldots \mathbf{b}_s),$$

it merely remains to observe that one or all of the three periods

r, s, $r+s$ must be even.* It follows (by induction) that *every product of an even [odd] number of transpositions is an even [odd] permutation.*

3·3. Groups. The subject of group-theory has been adequately expounded many times, so we shall be content to recall just the most relevant of its topics, in an attempt to make this book reasonably self-contained.

A set of elements or " operations " is said to form an *abstract group* if it is closed with respect to some kind of associative "multiplication", if it contains an "identity", and if each operation has an "inverse". More precisely, a group contains, for every two of its operations R and S, their product RS; 3·21 holds for all R, S, T; there is an identity, 1, such that

$$1\mathrm{R} = \mathrm{R}$$

for all R; and each R has an inverse, R^{-1}, such that

$$\mathrm{R}^{-1}\mathrm{R} = 1.$$

It is then easily deduced that $\mathrm{R}1 = \mathrm{R}$ and $\mathrm{R}\mathrm{R}^{-1} = 1$.

The number of distinct operations (including the identity) is called the *order* of the group. This is not necessarily finite.

A subset whose products (with repetitions) comprise the whole group is called a set of *generators* (as these operations " generate " the group). In particular, a single operation R generates a group which consists of all the powers of R, including $\mathrm{R}^0=1$. This is called a *cyclic* group; it is finite if R is periodic, and then its order is equal to the period of R. We may say that the cyclic group of order p is defined by the relation

$$\mathrm{R}^p = 1,$$

with the tacit understanding that $\mathrm{R}^n \neq 1$ for $0<n<p$. More generally, any group is defined by a suitable set of *generating relations*; e.g., the relations

3·31 $$\mathrm{R}_1{}^2 = \mathrm{R}_2{}^2 = (\mathrm{R}_1\,\mathrm{R}_2)^3 = 1$$

define a group of order 6 whose operations are 1, R_1, R_2, $\mathrm{R}_1\,\mathrm{R}_2$, $\mathrm{R}_2\,\mathrm{R}_1$, and $\mathrm{R}_1\,\mathrm{R}_2\,\mathrm{R}_1=\mathrm{R}_2\,\mathrm{R}_1\,\mathrm{R}_2$.

* This proof is taken from Levi **1**, p. 7. Note that Levi multiplies from right to left.

A subset which itself forms a group is called a *subgroup*. (For the sake of completeness it is customary to include among the subgroups the whole group itself and the group of order one consisting of 1 alone.) In particular, each operation of any group generates a cyclic subgroup.

If a given subgroup consists of T_1, T_2, . . . , while S is any operation in the group, the set of operations ST_i is called a *left coset* of the subgroup, and the set T_iS is called a *right coset*.* It can be proved that any two left (or right) cosets have either the same members or entirely different members. Hence the subgroup effects a distribution of all the operations in the group into a certain number of entirely distinct left (or right) cosets. This number is called the *index* of the subgroup. When the group is finite, the index is the quotient of the orders of the group and subgroup.

Two operations T and T′ are said to be *conjugate* if one can be transformed into the other, i.e., if the group contains an operation S such that $T'=T^S$, or $ST'=TS$. The relation of conjugacy is easily seen to be reflexive, symmetric, and transitive. A subgroup T_1, T_2, . . . is said to be *self-conjugate* if, for every S in the group, the operations T_i are a permutation of their transforms T_i^S, i.e., if the left and right cosets ST_i and T_jS are identical (apart from order of arrangement of members). In particular, any subgroup of index 2 is self-conjugate.

If two groups, G_1 and G_2, have no common operations except the identity, and if each operation of G_1 commutes with each operation of G_2, then the group generated by G_1 and G_2 is called their *direct product*, $\mathsf{G}_1 \times \mathsf{G}_2$. (This clearly contains G_1 and G_2 as self-conjugate subgroups.) For instance, the cyclic group of order pq, where p and q are co-prime, is the direct product of cyclic groups of orders p and q (generated by R^q and R^p, if R generates the whole group).

When the operations are interpreted as transformations, we have a representation of the abstract group as a *transformation group*. Since transformations automatically satisfy 3·21, we may say that a set of transformations forms a group if it contains the inverse of each member and the product of each pair. In particular, a group may consist of certain permutations of n objects ; it is then called a permutation group of degree n. A permutation group is said to be *transitive* (on the n objects) if its operations

* Birkhoff and MacLane 1, p. 146.

suffice to replace one object by all the others in turn. The three most important transitive groups are :

(i) the symmetric group of order $n!$, which consists of *all* the permutations of the n objects,

(ii) the alternating group of order $n!/2$, which consists of the *even* permutations,

(iii) the cyclic group of order n, which consists of the *cyclic* permutations, viz., the powers of the cycle $(\mathbf{a}_1 \ldots \mathbf{a}_n)$.

We easily verify that the alternating group is a subgroup of index 2 in the symmetric group (of the same degree). When $n=2$, (i) and (iii) are the same. When $n=3$, (ii) and (iii) are the same.

The six operations of the symmetric group on **a**, **b**, **c** are

3·32 $$1,\quad (\mathbf{a}\,\mathbf{b}),\quad (\mathbf{a}\,\mathbf{c}),\quad (\mathbf{b}\,\mathbf{c}),\quad (\mathbf{a}\,\mathbf{b}\,\mathbf{c}),\quad (\mathbf{a}\,\mathbf{c}\,\mathbf{b}).$$

In terms of the two generators $R_1=(\mathbf{a}\,\mathbf{b})$ and $R_2=(\mathbf{a}\,\mathbf{c})$, these are

$$1,\quad R_1,\quad R_2,\quad R_1 R_2 R_1,\quad R_1 R_2,\quad R_2 R_1.$$

It is instructive to compare this with the group consisting of the following six transformations of a variable x :

$$x'=x,\quad x'=1-x,\quad x'=\frac{1}{x},\quad x'=\frac{x}{x-1},\quad x'=\frac{1}{1-x},\quad x'=\frac{x-1}{x}.$$

Two such groups are said to be *isomorphic*, because they have the same " multiplication table " and consequently both represent the same *abstract* group.* In the present instance the abstract group is defined by 3·31.

Let a group G contain a self-conjugate subgroup T. Then any operation S of G occurs in a definite coset $\langle S\rangle = S\mathsf{T} = \mathsf{T}S$. The distinct cosets can be regarded as the operations of another group, in which products, identity, and inverse are defined by

$$\langle R\rangle \langle S\rangle = \langle RS\rangle,\quad \langle 1\rangle = \mathsf{T},\quad \langle S\rangle^{-1} = \langle S^{-1}\rangle.$$

This new group is called a *factor group* of G, or more explicitly the quotient group G/T. If it is finite, its order is equal to the index of T in G.

It may happen that G contains a subgroup S whose operations

* For an interesting discussion of the identification of isomorphic systems, see Levi **1**, p. 70.

S_j "represent" the cosets of T, in the sense that the distinct cosets are precisely $\langle S_j \rangle$. Then S is isomorphic with G/T. For instance, if G is the symmetric group 3·31, while T is the cyclic subgroup generated by $R_1 R_2$, then S could consist of 1 and R_1. Again, if G is the continuous group of all displacements, while G/T is the same group regarded as "operating on bundles of parallel rays" (see page 38), then T is the group of all translations, and S is the group of rotations leaving one point invariant.

It may happen, further, that the subgroup S is self-conjugate, like T. Then $T_i S_j = S_j T_i$, and $\mathsf{G} = \mathsf{S} \times \mathsf{T}$. For instance, if G is the cyclic group of order 6 defined by $R^6 = 1$, S and T might be the cyclic subgroups generated by R^2 and R^3, respectively.

3·4. Symmetry operations. When we say that a figure is "symmetrical", we mean that there is a congruent transformation which leaves it unchanged as a whole, merely permuting its component elements. For instance, when we say (as in § 2·8) that a zonohedron has central symmetry, we mean that there is an inversion which leaves it invariant. Such a congruent transformation is called a *symmetry operation.* Clearly, all the symmetry operations of a figure together form a group (provided we include the identity). This is called the *symmetry group* of the figure.

Conversely, given a group of congruent transformations, we can construct a symmetrical figure by taking all the transforms of any one point. The group is a subgroup of the symmetry group of the figure; in fact, it is usually the whole symmetry group. If the given group is finite, the figure consists of a finite number of points which the transformations permute. These points have a centroid (or "centre of gravity") which is transformed into itself. Thus

3·41. *Every finite group of congruent transformations leaves at least one point invariant.**

It follows that the transforms of any point by such a group lie on a sphere.

A group of transformations may be *discrete* without being finite. This means that every point has a discrete set of transforms, i.e., that any given point has a neighbourhood containing none of its transforms (save the given point itself).

* Bravais 1, p. 143 (Théorème III).

In the case of the cyclic group generated by a single congruent transformation S, the transforms of a point $\mathbf{A}_0$ of general position are

$$\ldots, \mathbf{A}_{-2}, \mathbf{A}_{-1}, \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \ldots,$$

where $\mathbf{A}_n = \mathbf{A}_0{}^{S^n}$. These may be regarded as the vertices of a generalized *regular polygon* (cf. § 1·1).

The various kinds of congruent transformation lead to various kinds of polygon. If S is a reflection, half-turn, or inversion, the polygon reduces to a digon, $\{2\}$. If S is a rotation, the sides are equal chords of a circle ; if the angle of rotation is $2\pi/p$, we have the ordinary regular polygon, $\{p\}$. (The case where p is rational but not integral will be developed in § 6·1.) If S is a translation we have the limiting case where p becomes infinite : a sequence of equal segments of one line, the *apeirogon*, $\{\infty\}$. If S is a glide-reflection, the "polygon" is a plane zigzag. If S is a rotatory-reflection, it is a *skew* zigzag, whose vertices lie alternately on two equal circles in parallel planes ; if the angle of the component rotation is π/p, the sides are the lateral edges of a p-gonal anti-prism. (Cases where $p=2$, 3, 5 occurred as Petrie polygons in § 2·6.) Finally, if S is a screw-displacement we have a *helical* polygon, whose sides are equal chords of a helix.

In every case except that of the digon, the cyclic group generated by S is *not* the whole symmetry group of the generalized polygon ; e.g., there is a symmetry operation interchanging $\mathbf{A}_n$ and $\mathbf{A}_{-n}$ for all values of n (simultaneously). In the case of the ordinary polygon $\{p\}$, the line joining the centre to any vertex, or to the mid-point of any side, contains one other vertex or mid-side point ; thus there are p such lines. The p-gon is symmetrical by a half-turn about any of them, besides being symmetrical by rotation through any multiple of $2\pi/p$ about the "axis" of the polygon. Thus the complete symmetry group of $\{p\}$ is of order $2p$, consisting of p half-turns about concurrent lines in the plane of the polygon, and p rotations through various angles about one line perpendicular to that plane.

The symmetry operations of a figure are either all direct, or half direct and half opposite. For, if an opposite operation occurs, its products with all the direct operations are all the opposite operations. Thus the *rotation group* formed by the direct operations is either the whole symmetry group or a subgroup of index 2. In the latter case the opposite operations form the single distinct coset of this subgroup.

The complete symmetry group of $\{p\}$, as described above, is the rotation group of the dihedron $\{p, 2\}$ (§ 1·7), and is consequently known as the *dihedral group* of order $2p$. On the other hand, the complete symmetry group of $\{p, 2\}$ is of order $4p$, as it contains also the same rotations multiplied by the reflection that interchanges the two faces of the dihedron. As a symmetry operation of $\{p\}$ itself, the reflection in its own plane does not differ from the identity. Thus the p half-turns can be replaced by their products with this reflection, which are reflections in p coaxial planes.

The situation becomes clearer when we take a purely two-dimensional standpoint, considering rotations about *points* and reflections in *lines*. Then the symmetry group of $\{p\}$ consists of p reflections (in lines joining the centre to the vertices and mid-side points) and p rotations (about the centre); but the *rotation* group of $\{p\}$ is cyclic.

It is interesting to observe that the dihedral group of order 6 (or " trigonal dihedral group ") is isomorphic with the symmetric

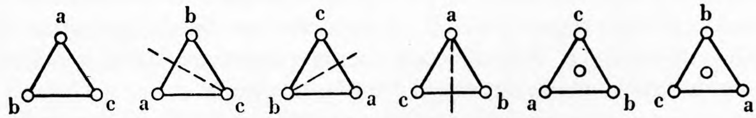


FIG. 3·4A

group of degree 3. In fact, the six symmetry operations of the equilateral triangle $\{3\}$ permute the vertices **a**, **b**, **c** in accordance with 3·32 (see Fig. 3·4A). The transpositions appear as reflections, and the cyclic permutations as rotations.

3·5. The polyhedral groups. The most interesting finite groups of rotations are the rotation groups of the regular polyhedra, which we proceed to investigate.

Every rotation that occurs in a finite group is of finite period; so its angle must be commensurable with π. In fact, the smallest angle of rotation about a given axis is a submultiple of 2π, and all other angles of rotation about the same axis are multiples of this smallest one. For,* if j and p are co-prime, we can find a multiple of j/p which differs from $1/p$ by an integer; so if $2\pi j/p$ is the smallest angle of rotation that occurs, we must have $j=1$. The rotations about this axis then form a cyclic group of order

* Bravais 1, p. 142.

p, so we speak of an *axis of p-fold rotation.* When $p=2$, 3, 4, or 5, the axis is said to be digonal, trigonal, tetragonal, or pentagonal.

Two reciprocal polyhedra obviously have the same symmetry group, and likewise the same rotation group. The centre of $\{p, q\}$ is joined to the vertices, mid-edge points, and centres of faces, by axes of q-fold, 2-fold, and p-fold rotation. Clearly, no further axes of rotation can occur. In other words, the direct symmetry operations of the polyhedron consist of rotations through angles $2k\pi/q$, π, and $2j\pi/p$, about these respective lines. If we exclude the identity, these rotations involve $q-1$ values for k, and $p-1$ for j. But the vertices, mid-edge points, and face-centres occur in antipodal pairs. (In the case of the tetrahedron, each vertex is opposite to a face.) Hence the total number of rotations, excluding the identity, is

$$\tfrac{1}{2}[N_0(q-1)+N_1+N_2(p-1)]=\tfrac{1}{2}(N_0q-2+N_2p)=2N_1-1$$

(by 1·61 and 1·71), and *the order of the rotation group is* $2N_1$.

The same result may also be seen as follows. Let a sense of direction be assigned to a particular edge. Then a rotational symmetry operation is determined by its effect on this directed edge. Thus there is one such rotation for each edge, directed in either sense: $2N_1$ rotations altogether. Still more simply, the order of the rotation group is equal to the number of edges of $\left\{\begin{matrix}p\\q\end{matrix}\right\}$; for the group is transitive on those edges, and the subgroup leaving one of them invariant is of order 1.

In particular, we have the *tetrahedral* group of order 12, the *octahedral* group of order 24 (which is also the rotation group of the cube) and the *icosahedral* group of order 60 (which is also the rotation group of the dodecahedron). In § 3·6 we shall identify these with permutation groups of degree 4 or 5.

3·6. The five regular compounds. We define a compound polyhedron (or, briefly, a *compound*) as a set of equal regular polyhedra with a common centre. The compound is said to be *vertex-regular* if the vertices of its components are together the vertices of a single regular polyhedron, and *face-regular* if the face-planes of its components are the face-planes of a single regular polyhedron. For instance, the diagonals of the faces of a cube are the edges of two reciprocal tetrahedra. (See Plate I, Fig. 6, or Plate III, Fig. 5.) These form a compound, Kepler's *stella octangula*, which is both vertex-regular and face-regular : its vertices belong to a cube, and its face-planes to an octahedron.

We shall find it convenient to have a definite notation for compounds.* If d distinct $\{p, q\}$'s together have the vertices of $\{m, n\}$, each counted c times, or the faces of $\{s, t\}$, each counted e times, or both, we denote the compound by

$$c\{m, n\}[d\{p, q\}] \quad \text{or} \quad [d\{p, q\}]e\{s, t\} \quad \text{or} \quad c\{m, n\}[d\{p, q\}]e\{s, t\}.$$

The reciprocal compound is clearly

$$[d\{q, p\}]c\{n, m\} \quad \text{or} \quad e\{t, s\}[d\{q, p\}] \quad \text{or} \quad e\{t, s\}[d\{q, p\}]c\{n, m\}.$$

The numbers of vertices of $\{m, n\}$ and $\{p, q\}$ are in the ratio $d : c$, and the numbers of faces of $\{s, t\}$ and $\{p, q\}$ are in the ratio $d : e$. For instance, the *stella octangula* is

$$\{4, 3\}[2\{3, 3\}]\{3, 4\}$$

(with $c=e=1$). Other examples will be obtained in the course of the following investigation of the polyhedral groups.

In order to identify the tetrahedral group with the alternating group of degree 4, we observe that the vertices of a regular tetrahedron are four points whose six mutual distances are all equal.

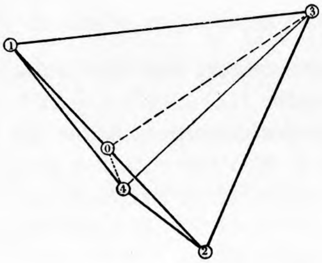


FIG. 3·6A

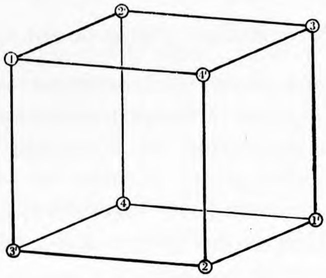
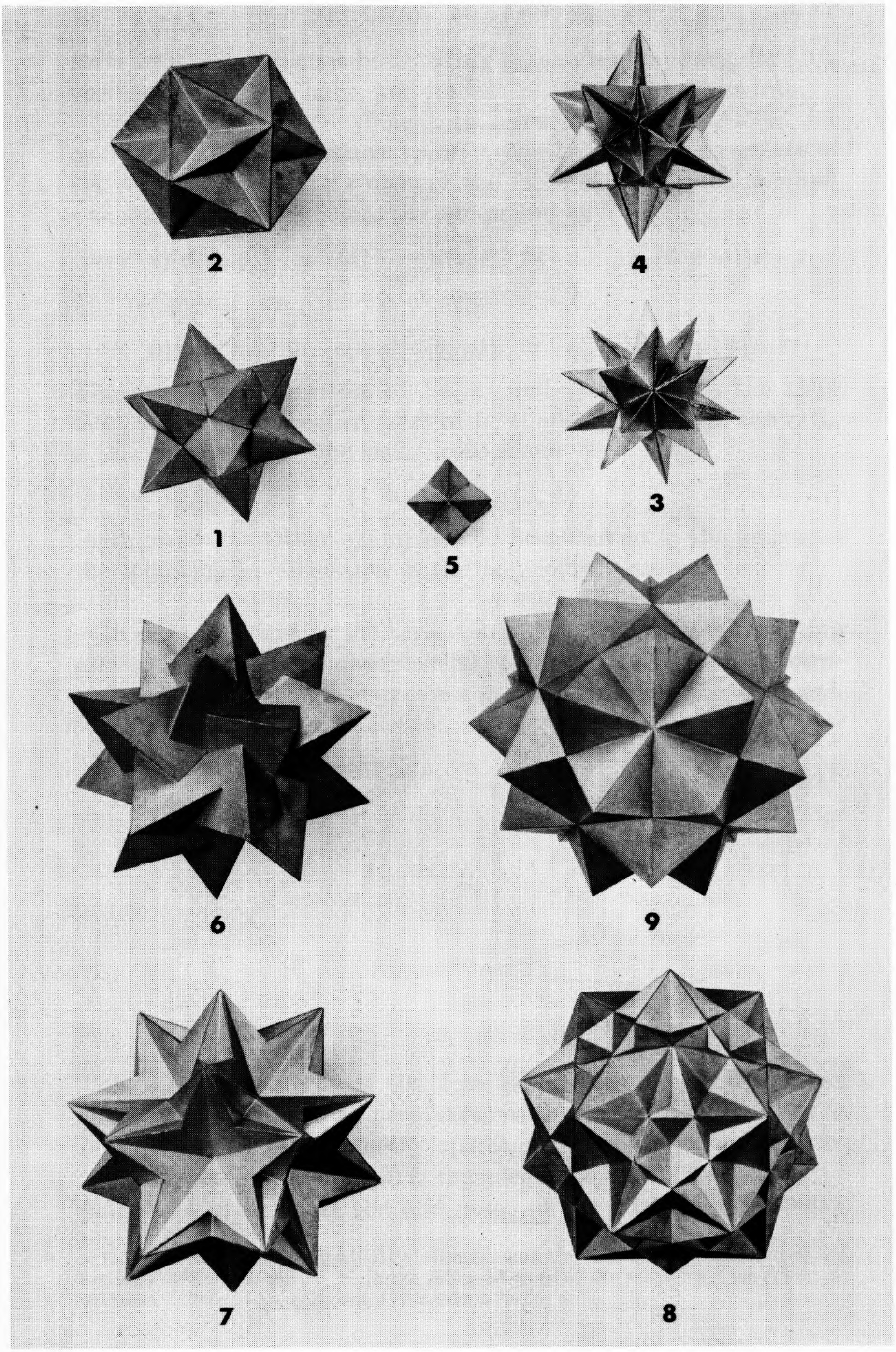


FIG. 3·6B

This statement involves the four points symmetrically, so we should expect all the 24 permutations in the symmetric group to be represented by symmetry operations of the tetrahedron. In fact, the transposition (**1 2**) is represented by the reflection in the plane **034**, where **0** is the mid-point of the edge **12** (Fig. 3·6A).

* This symbolism is admittedly clumsy, but the obvious alternatives would be more difficult to print. Note the different roles of the numbers c (or e) and d : we have d distinct $\{p, q\}$'s, but c coincident $\{m, n\}$'s.



REGULAR STAR-POLYHEDRA AND COMPOUNDS

But any *even* permutation, being the product of *two* transpositions, is represented by a rotation. Thus the "tetrahedral group" (which we have defined as consisting of rotations alone) is the *alternating* group of degree 4.

In the *stella octangula*, every symmetry operation of either tetrahedron is also a symmetry operation of the cube; but the cube has additional operations which interchange the two tetrahedra. The rotation group of the tetrahedron **1234** evenly permutes the four diameters **11′**, **22′**, **33′**, **44′** of the cube (Fig. 3·6B). But the *odd* permutations of these diameters likewise occur as rotations; e.g., **11′** and **22′** are transposed by a half-turn about the join of the mid-points of the two edges **12′** and **21′**. Hence the octahedral group (which is the rotation group of the cube) is the *symmetric* group of degree 4.

If **ABCDE** and **AEFGH** are two adjacent faces of a regular dodecahedron, the vertices **BDFH** clearly form a square, whose sides join alternate vertices of pentagons. Moreover, these four vertices, with their antipodes, form a cube; and alternate vertices of this cube form a tetrahedron (such as **1111** in Fig. 3·6C or D). It is easily seen that the rotations of this tetrahedron into itself are symmetry operations of the whole dodecahedron, i.e., that the tetrahedral group occurs as a subgroup in the icosahedral group (as well as in the octahedral group). The remaining operations of the icosahedral group transform this tetrahedron into others of the same sort, making altogether a compound of *five tetrahedra* inscribed in the dodecahedron. (Plate III, Fig. 6.) In other words, the twenty vertices of the dodecahedron are distributed in sets of four among five tetrahedra. The central inversion transforms this into a second compound of five tetrahedra, enantiomorphous (and reciprocal) to the first. The two together form a compound of *ten tetrahedra* (Fig. 7), reciprocal pairs of which can be replaced by *five cubes* (Fig. 8). Here each vertex of the dodecahedron belongs to two of the tetrahedra, and to two of the cubes.

We have thus obtained three vertex-regular compounds whose vertices belong to a dodecahedron. By reciprocation, we find that the compounds of tetrahedra are also face-regular, their face-planes belonging to an icosahedron. But the face-planes of the five cubes belong to a triacontahedron, so the reciprocal is a face-regular compound of *five octahedra* whose vertices belong to an icosidodecahedron. (Plate III, Fig. 9.) The appropriate symbols are :

$$\{5, 3\}[5\{3, 3\}]\{3, 5\},$$
$$2\{5, 3\}[10\{3, 3\}]2\{3, 5\},$$
$$2\{5, 3\}[5\{4, 3\}] \quad \text{and} \quad [5\{3, 4\}]2\{3, 5\}.$$

A very pretty effect is obtained by making models of these compounds, with a different colour for each component. The colouring of the five cubes determines a colouring of the triacontahedron in five colours, so that each face and its four neighbours have different colours. This scheme is used by Kowalewski as an aid to his *Bauspiel* (see § 2·7).

The two enantiomorphous compounds of five tetrahedra may be distinguished as *laevo* and *dextro*. They provide a convenient symbolism for the twenty vertices of the dodecahedron (or for the twenty faces of the icosahedron) as follows. We number the five tetrahedra of the *laevo* compound as in Fig. 3·6C ; those of the *dextro* compound (Fig. 3·6D) acquire the same numbers by means of the central inversion. Then the vertices of the dodecahedron are denoted by the twenty ordered pairs **12**, **21**, **13**, **31**, . . . , **45**, **54**, in such a way that ij is a vertex of the ith *laevo* tetrahedron and of the jth *dextro* tetrahedron. (For simplicity, the dodecahedron in Fig. 3·6E has been drawn as an opaque solid. The symbols for the hidden vertices are easily supplied, as ji is antipodal to ij.)

Each direct symmetry operation of the dodecahedron is representable as a permutation of the five digits ; e.g., the permutation (**1 2 3**) is a trigonal rotation about the diameter joining the opposite vertices **45** and **54**, (**1 4**)(**3 5**) is a digonal rotation about the join of the mid-points of edges **13 45** and **31 54**, and (**1 2 3 4 5**) is a pentagonal rotation about the join of centres of two opposite faces. Since all these are *even* permutations, we have proved that *the icosahedral group is the alternating group of degree* 5.

To sum up, the symmetric groups of degrees 3 and 4 are the rotation groups of $\{3, 2\}$ and $\{3, 4\}$, and the alternating groups of degrees 4 and 5 are the rotation groups of $\{3, 3\}$ and $\{3, 5\}$.

3·7. Coordinates for the vertices of the regular and quasi-regular solids. The only regular polyhedron whose faces can be coloured alternately white and black, like a chess board, is the octahedron $\{3, 4\}$. For, this is the only polyhedron $\{p, q\}$ with q even. In Fig. 3·6B we denoted the vertices of the cube by

1, 2, 3, 4, 1′, 2′, 3′, 4′. By reciprocation, the same symbols can be used for the faces of the octahedron, and we may distinguish the two sets of four faces as white and black. (In other words, we colour the faces of the octahedron like those of a *stella octangula* whose two tetrahedra are white and black, respectively.) By assigning a clockwise sense of rotation to each white face, and a counterclockwise sense to each black face, we obtain a *coherent*

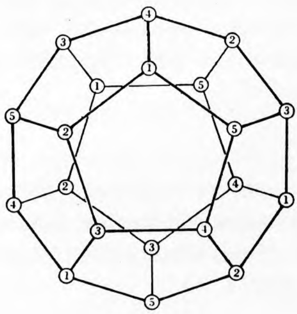


Fig. 3·6c

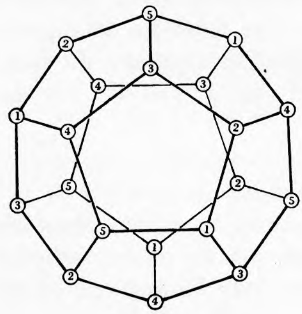


Fig. 3·6d

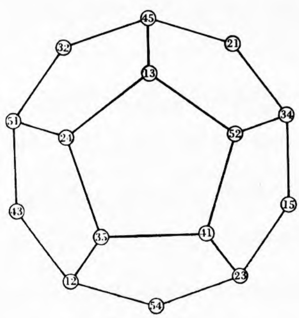


Fig. 3·6e

indexing of the edges, such as can be indicated by marking an arrow along each edge. Then, if we proceed along an edge in the indicated direction, there will be a white face on our right side and a black face on our left.

This enables us to define, for any given ratio $a : b$, twelve points dividing the respective edges in this ratio, so that the three points in each face form an equilateral triangle. In general, these twelve points will be the vertices of an irregular icosahedron, whose faces consist of eight such equilateral triangles and twelve isosceles triangles. Without loss of generality, we may suppose that $a \geqslant b$. When a/b is large, the isosceles triangles have short bases; in the limit they disappear, as their equal sides coincide and lie along the twelve edges of the octahedron. But when a approaches equality with b, the isosceles triangles tend to become right-angled; in the limit, pairs of them form halves of the six square faces of a cuboctahedron, as in Fig. 8·4A on page 152.* By considerations of continuity, we see that at some intermediate stage the isosceles triangles must become equilateral, and the icosahedron regular. In fact, the squares of the respective sides are a^2-ab+b^2 and $2b^2$, which are equal if

$$a^2 - ab - b^2 = 0,$$

so that $a/b=\tau$. (See 2·47.) Thus *the twelve vertices of the icosahedron can be obtained by dividing the twelve edges of an octahedron according to the golden section.*† For a given icosahedron, the octahedron may be any one of the $[5\{3, 4\}]2\{3, 5\}$.

In terms of rectangular Cartesian coordinates, the vertices of a cube (of edge 2) are

3·71 $$(\pm 1,\ \pm 1,\ \pm 1),$$

those of a tetrahedron (of edge $2\sqrt{2}$) are

3·72 $$(1, 1, 1),\ (1, -1, -1),\ (-1, 1, -1),\ (-1, -1, 1),$$

and those of an octahedron (of edge $\sqrt{2}$) are

3·73 $$(\pm 1, 0, 0),\ (0, \pm 1, 0),\ (0, 0, \pm 1).$$

If $a+b=1$, the segment joining (0, 0, 1) and (0, 1, 0) is divided in the ratio $a : b$ by the point $(0, a, b)$. Such points on all the edges of the octahedron 3·73 are

$$(0, \pm a, \pm b),\ (\pm b, 0, \pm a),\ (\pm a, \pm b, 0).$$

Hence the vertices of a cuboctahedron (of edge $\sqrt{2}$) are

3·74 $$(0, \pm 1, \pm 1),\ (\pm 1, 0, \pm 1),\ (\pm 1, \pm 1, 0),$$

and the vertices of an icosahedron (of edge 2) are

3·75 $$(0, \pm \tau, \pm 1),\ (\pm 1, 0, \pm \tau),\ (\pm \tau, \pm 1, 0).$$

* See also Coxeter **13**, p. 396.

† Cf. Schönemann **1**.

The planes of the faces

$$(0, \tau, 1)\ (\pm 1, 0, \tau) \quad \text{and} \quad (0, \tau, 1)\ (1, 0, \tau)\ (\tau, 1, 0)$$

are respectively

$$\tau^{-1}y + \tau z = \tau^2 \quad \text{and} \quad x + y + z = \tau^2.$$

(Remember that $\tau^2 = \tau + 1$.) Hence the vertices of the reciprocal dodecahedron (of edge $2\tau^{-1}$) are

3·76 $(0, \pm\tau^{-1}, \pm\tau)$, $(\pm\tau, 0, \pm\tau^{-1})$, $(\pm\tau^{-1}, \pm\tau, 0)$, $(\pm 1, \pm 1, \pm 1)$.

One of the five inscribed cubes is thus made very evident.

The mid-point of the edge $(\tau, \pm 1, 0)$ of the icosahedron 3·75 is $(\tau, 0, 0)$, while that of the edge $(1, 0, \tau)\ (\tau, 1, 0)$ is $(\frac{1}{2}\tau^2, \frac{1}{2}, \frac{1}{2}\tau)$. Hence (after multiplication by $2\tau^{-1}$) the vertices of an icosidodecahedron (of edge $2\tau^{-1}$) are

3·77 $$\begin{cases} (\pm 2, 0, 0), & (0, \pm 2, 0), & (0, 0, \pm 2), \\ (\pm\tau, \pm\tau^{-1}, \pm 1), & (\pm 1, \pm\tau, \pm\tau^{-1}), & (\pm\tau^{-1}, \pm 1, \pm\tau). \end{cases}$$

The vertices in the upper row belong to one of the octahedra of $[5\{3, 4\}]2\{3, 5\}$.

3·8. The complete enumeration of finite rotation groups. In §§ 3·4 and 3·5 we considered various groups of rotations : cyclic, dihedral, tetrahedral, octahedral, icosahedral. The question now arises, Are these the *only* finite groups of rotations ? If so, they are also the only finite groups of *displacements* (by 3·41 and 3·12). We shall find that the answer is Yes.

Consider the general finite group of rotations. Since there is an invariant point **O** (lying on the axes of all the rotations), it is convenient to regard the group as operating on a *sphere* with centre **O**, instead of the whole space. Each rotation, having for axis a diameter of the sphere, is then regarded as a rotation about a *point* on the sphere. (We must remember, however, that the rotation through angle θ about any point is the same as the rotation through $-\theta$ about the antipodal point.) We saw (as a consequence of 3·12) that the product of two such rotations is another. To determine the product of two given rotations, we make use of the following theorem :

If the vertices of a spherical triangle **PQR** (like the triangle

$\mathbf{PQ_1R}$ in Fig. 3·8A) are named in the negative (or clockwise) sense, *the product of rotations through angles* $2P$, $2Q$, $2R$ *about* **P**, **Q**, **R** *is the identity.*

To prove this, we merely have to express the given product of rotations as the product of reflections in the great circles **RP**, **PQ** ; **PQ**, **QR** ; **QR**, **RP**.

It follows that the product of rotations through $2P$ and $2Q$ about **P** and **Q** is the rotation through $-2R$ about **R**. In particular, the product of half-turns about any two points **P** and **Q** is the rotation through $-2\angle\mathbf{POQ}$ about one of the poles of the great circle **PQ** (or through $+2\angle\mathbf{POQ}$ about the other pole). This product of half-turns cannot itself be a half-turn unless the axes **OP** and **OQ** are perpendicular. Hence, if a rotation group has no operation of period greater than 2, it must be either the group of order 2 generated by a single half-turn, or the "four-group" generated by two half-turns about perpendicular axes; i.e., it must either be the cyclic group of order 2 or the dihedral group of order 4.

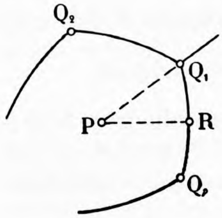


FIG. 3·8A

Secondly, if there is just one axis of p-fold rotation where $p>2$, this must be perpendicular to any digonal axes that may occur. Hence the group is either cyclic of order p or dihedral of order $2p$.

Finally, if there are several axes of more than 2-fold rotation, let one of them be **OP**, so that there is a rotation through $2\pi/p$ about **P**. The group being finite, there is a least distance from **P** (on the sphere) at which we can find a point $\mathbf{Q_1}$ lying on another axis of more than 2-fold rotation, say q-fold rotation. Successive rotations through $2\pi/p$ about **P** transform $\mathbf{Q_1}$ into other centres of q-fold rotation, say $\mathbf{Q_2}, \ldots, \mathbf{Q_p}$, lying on a small circle within which **P** is the *only* centre of more than 2-fold rotation. (See Fig. 3·8A.) The product of rotations through $2\pi/p$ and $2\pi/q$ about **P** and $\mathbf{Q_1}$ is the rotation through $-2\pi/r$ about a point **R** such that the spherical triangle $\mathbf{PQ_1R}$ has angles π/p, π/q, π/r.

We proceed to determine the position of **R** and the value of r. (We cannot yet say whether r is an integer.) Since the angle-sum of any spherical triangle is greater than π, we have

3·81 $$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1.$$

But $p \geqslant 3$ and $q \geqslant 3$. Hence $r<3$, and consequently $q>r$. Thus the triangle $\mathbf{PQ_1R}$ has a smaller angle at $\mathbf{Q_1}$ than at $\mathbf{R}$, and the same inequality must hold for the respectively opposite sides. Hence $\mathbf{R}$ lies inside the small circle around $\mathbf{P}$, and $\mathbf{OR}$ must be a *digonal* axis; so the rotation through $-2\pi/r$ about $\mathbf{R}$, which transforms $\mathbf{Q}_p$ into $\mathbf{Q}_1$, can only be a half-turn. Hence $r=2$, and $\mathbf{OR}$ bisects the angle $\mathbf{Q}_p\mathbf{OQ}_1$, i.e., $\mathbf{R}$ is the mid-point of the side $\mathbf{Q}_p\,\mathbf{Q}_1$ of the spherical p-gon $\mathbf{Q}_1\,\mathbf{Q}_2 \ldots \mathbf{Q}_p$. Successive rotations through $2\pi/q$ about $\mathbf{Q}_1$ transform this p-gon (of angle $2\pi/q$) into a set of q p-gons completely surrounding their common vertex $\mathbf{Q}_1$. Further rotations of the same kind lead to a number of p-gons fitting together to cover the whole sphere.

Thus the transforms of $\mathbf{Q}_1$ are the vertices of the regular polyhedron $\{p, q\}$, the transforms of $\mathbf{P}$ are the vertices of the reciprocal polyhedron $\{q, p\}$, and the transforms of $\mathbf{R}$ are the vertices of the "semi-reciprocal" polyhedron $\left\{ \begin{matrix} p \\ q \end{matrix} \right\}$. The inequality 3·81 (or 2·32) reduces to 1·73, and we have the three polyhedral groups of § 3·5. (The triangle $\mathbf{PQ_1R}$ was called $\mathbf{P}_2\,\mathbf{P}_0\,\mathbf{P}_1$ in § 2·5.)

From our construction we can be sure that the p-gonal and q-gonal axes through the vertices of $\{q, p\}$ and $\{p, q\}$ are the only axes of more than 2-fold rotation. But can we be sure that the axes through the vertices of $\left\{ \begin{matrix} p \\ q \end{matrix} \right\}$ are the only digonal axes? Might not a further digonal axis occur midway between $\mathbf{P}$ and $\mathbf{Q}_1$, if $p=q$? No: that half-turn would combine with the rotation through $2\pi/p$ about $\mathbf{P}$ to give a rotation of period 4 about $\mathbf{R}$, which is absurd.

3·9. Historical remarks. The kinematics of a rigid body (§ 3·1) was founded by Euler (1707-1783) and developed by Chasles, Rodrigues, Hamilton, and Donkin. In particular, 3·12 is commonly called "Euler's Theorem".

The theory of permutation groups (or "substitution groups") was developed by Lagrange (1736-1813), Ruffini, Abel (1802-1829), Galois (1811-1832), Cauchy (1789-1857), and Jordan (whose famous *Traité des Substitutions* appeared in 1870). Lagrange proved that the order of a group is divisible by the order of any subgroup. Galois made such important contributions to the subject that he eventually became recognized as the real founder of group-theory; yet his contemporaries scorned him, and he was murdered* at the age of twenty. The notion of a self-conjugate subgroup is due to him, and it was he who first distributed the operations of a group into cosets (though the actual word " co-set " was coined in 1910 by G. A. Miller). The first precise definition of an abstract group was given in 1854 by Cayley (1).

In § 3·4 we considered the set of transforms of a single point by a group of congruent transformations. This idea occurs in a posthumous paper of Möbius (2). The rotation group of the regular polyhedron $\{p, q\}$ was investigated in 1856 by Hamilton (1), who gave an abstract definition equivalent to

$$\mathrm{R}^p = \mathrm{S}^q = (\mathrm{RS})^2 = 1.$$

The polyhedral groups also arose in the work of Schwarz and Klein, as groups of transformations of a complex variable. The first chapter of the latter's *Lectures on the Icosahedron* (Klein 2) may well be read concurrently with §§ 3·5 and 3·6.

The compound polyhedra were thoroughly investigated by Hess in 1876.† But the *stella octangula* $\{4, 3\}[2\{3, 3\}]\{3, 4\}$ had already been discovered by Kepler (1, p. 271) and may almost be said to have been anticipated in Euclid XV, 1 and 2. It occurs in nature as a crystal-twin of tetrahedrite: The existence of the remaining compounds is a simple consequence of Kepler's observation that a cube can be inscribed in a dodecahedron. It was Hess who first gave Cartesian coordinates for the vertices of all the regular and quasi-regular polyhedra,‡ as in § 3·7.

We proved in § 3·8 that the only finite groups of rotations are the cyclic, dihedral, and polyhedral groups. Our proof is essentially that of Bravais, amplified by justifying his assumption ‖ " Le point A viendra en C." Bravais's proof occurs as part of the more complicated problem of enumerating the finite groups of congruent transformations, which includes the enumeration of the 32 geometrical crystal classes.** This enumeration was

* See Infeld 1.

† Hess 1, pp. 39 (five octahedra), 45 (five or ten tetrahedra), 52 and 68 (five cubes). Klein (1, p. 19) remarks in a footnote that " one sees occasionally (in old collections) models of 5 cubes which intersect one another in such a way. . . ."

‡ Hess 3, pp. 295, 340-343. For the *regular* polyhedra, see also Schoute 6, pp. 155-159.

‖ Bravais 1, p. 166. For this amplification I am indebted to Patrick Du Val. For a quite different approach, see Ford 1, p. 133 or Zassenhaus, 1, pp. 15-18. It is interesting to recall that Bravais (at the age of 18) won the prize in the General Competition, on the occasion when Galois was ranked fifth !

** Swartz 1, pp. 385-394; Burckhardt 2, p. 71.

first achieved in 1830, by Hessel (1), whose book remained unnoticed till 1891. The next step in the same direction was Sohncke's enumeration of 65 infinite discrete groups of displacements. Finally, after Pierre Curie had drawn attention to the importance of the rotatory-reflection, the famous enumeration of 230 infinite discrete groups of congruent transformations was made independently by Fedorov in Russia (1885), Schoenflies in Germany (1891), and Barlow in England (1894).