Covariant Differentiation.

Def. $\vec{V}(p): M \to \mathbb{R}^3$ is a smooth vector field on M if $\vec{V}(p) \in T_pM$ and $\vec{V}(x(u,v))$ is a smooth function of u, v.

We can differentiate $\vec{V}$ along M, but we only see the portion of the derivative in the tangent plane:

Def. The covariant derivative

$$\nabla_w V = \text{projection of } D_w V \in \mathbb{R}^3 \text{ onto } T_pM.$$
$$= D_w V - \langle D_w V, \vec{n} \rangle \vec{n}.$$

We can (similarly) differentiate vector fields defined only along a curve γ in M, but only in the γ' direction:

$$\nabla_{\gamma'(t)} V(\gamma(t)) = \frac{d}{dt} V(\gamma(t)) \text{ projected to } T_{\gamma(t)}M$$
$$= \frac{d}{dt} V(\gamma(t)) - \left\langle \frac{d}{dt} V(\gamma(t)), \vec{n} \right\rangle \vec{n}$$

A vector field is parallel along γ if $\nabla_{\gamma'(t)} V \equiv 0$.

Lemma. $\vec{V}$ is parallel along $\gamma \Leftrightarrow$ $\frac{d}{dt}\vec{V}(\gamma(t))$ is in the $\vec{n}$ direction.

Example. The tangent vector field and the field $(0,0,1)$ are both parallel along the great circle in x-y plane.

We can now consider an arbitrary vector field $\vec{V}(u,v) = a(u,v)\, X_u + b(u,v)\, X_v$, and work out the formula for covariant differentiation in the $\vec{w} = (w_u, w_v)$ direction. First, differentiating in the u direction

$$\nabla_{X_u} \vec{V} = a_u X_u + a X_{uu} + b_u X_v + b X_{vu}$$

$$= a_u X_u + a\left(\Gamma^u_{uu} X_u + \Gamma^v_{uu} X_v\right)$$
$$+ b_u X_v + b\left(\Gamma^u_{uv} X_u + \Gamma^u_{uv} X_v\right)$$

$$= \left(a_u + a\Gamma^u_{uu} + b\Gamma^u_{uv}\right) X_u + \left(b_u + a\Gamma^v_{uu} + b\Gamma^v_{uv}\right) X_v$$

(Notice that we just had to ignore the stuff in the $\vec{n}$ direction!) Similarly,

$$\nabla_{X_v} \vec{V} = \left(a_v + a\Gamma^u_{uv} + b\Gamma^u_{vv}\right) X_u$$
$$+ \left(b_v + a\Gamma^v_{uv} + b\Gamma^v_{vv}\right) X_v.$$

We can use this to differentiate in an arbitrary direction $\vec{\omega} = (\omega_u, \omega_v)$
$$= \omega_u X_u + \omega_v X_v$$

$$\nabla_{\vec{\omega}} \vec{V} = \omega_u\left(\nabla_{X_u} \vec{V}\right) + \omega_v\left(\nabla_{X_v} \vec{V}\right).$$

In fact, given an initial vector V_0 at $\gamma(0)$, $\exists$ a <u>unique</u> parallel vector field ~~solving~~ $V(t)$ solving

$$\nabla_{\gamma'(t)} V(t) \equiv 0, \qquad V(0) = V_0.$$

Now this equation will be important enough for us to want to work out the details from the definition:

$$\nabla_{\gamma'(t)} V(\gamma(t)) = \frac{d}{dt}\Big(a(t)\, X_u(u(t),v(t)) + b(t) X_v(u(t),v(t))\Big)^{||}$$

where $v^{||} = v - \langle v, \vec{n}\rangle \vec{n}$ (the projection of v to the tangent plane T_pM). So let's differentiate:

$$\begin{aligned} &= a'(t)\, X_u + a(t) \frac{d}{dt}\big(X_u(u(t),v(t))\big)^{||} \\ &\quad + b'(t)\, X_v + b(t) \frac{d}{dt}\big(X_v(u(t),v(t))\big)^{||} \\ &= a'(t)\, X_u + a(t)\big(u'(t) X_{uu} + v'(t) X_{uv}\big)^{||} \\ &\quad + b'(t) X_v + b(t)\big(u'(t) X_{vu} + v'(t) X_{vv}\big)^{||} \end{aligned}$$

$$= a'(t)X_u + a(t)\Big(u'(t)\big(\Gamma^u_{uu}X_u + \Gamma^v_{uu}X_v\big) + v'(t)\big(\Gamma^u_{uv}X_u + \Gamma^v_{uv}X_v\big)\Big)$$

$$+ b'(t)X_v + b(t)\Big(u'(t)\big(\Gamma^u_{uv}X_u + \Gamma^v_{uv}X_v\big) + v'(t)\big(\Gamma^u_{vv}X_u + \Gamma^v_{vv}X_v\big)\Big)$$

$$= \Big(a'(t) + u'(t)\big(a(t)\Gamma^u_{uu} + b(t)\Gamma^u_{uv}\big) + v'(t)\big(a(t)\Gamma^u_{uv} + b(t)\Gamma^u_{vv}\big)\Big)X_u$$

$$+ \Big(b'(t) + u'(t)\big(a(t)\Gamma^v_{uu} + b(t)\Gamma^v_{uv}\big) + v'(t)\big(a(t)\Gamma^v_{uv} + b(t)\Gamma^v_{vv}\big)\Big)X_v$$

Proposition. Given a smooth curve $\gamma(t) = X(u(t),$ and a starting vector $V(0) = a(0)X_u + b(0)X_v$, $\exists!$ a parallel vector field $V(t)$.

Proof. Since u', v' and the Christoffel symbols are smooth functions of t, we can rewrite these equations as

$$a'(t) + f_{aa}(t)\,a(t) + f_{ab}(t)\,b(t) = 0$$
$$b'(t) + f_{ba}(t)\,a(t) + f_{bb}(t)\,b(t) = 0$$

which is solvable (uniquely) by our theorem on ODEs. $\square$

Example. Find a parallel vector field along a lesser circle $u = u_0$ of the sphere $X(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$.

~~Pf~~

The curve is $\gamma(t) = (u_0, t)$ and the Christoffel symbols are

$\Gamma^u_{uu} = \Gamma^v_{uu} = 0$, $\Gamma^u_{uv} = 0$, $\Gamma^v_{uv} = \cot u$, $\Gamma^u_{vv} = -\sin u \cos u$, $\Gamma^v_{vv} =$

So our ODEs become

$$a'(t) + 0(\quad) + 1(0 + b(t)(-\sin u \cos u)) = 0$$
$$b'(t) + 0(\quad) + 1(a(t)\cot u + 0) = 0$$

or since $u = u_0$ is a constant,

$$a'(t) = b(t) \sin u_0 \cos u_0$$
$$b'(t) = a(t) (-\cot u_0).$$

Now we can convert this system of first order diff. eqs. into a single second order ODE by differentiating

$$\begin{aligned} b''(t) &= a'(t)(-\cot u_0) \\ &= b(t) \sin u_0 \cos u_0 (-\cot u_0) \\ &= -\cos^2 u_0 \, b(t) = -K^2 b(t). \end{aligned}$$

so we know

$$b''(t) + K^2 b(t) = 0.$$

The solutions to this equation are in the form

$$\begin{aligned} b(t) &= C_1 \cos(Kt) + C_2 \sin(Kt). \\ &= C_1 \cos((\cos u_0) t) + C_2 \sin((\cos u_0) t). \\ &= C \cos(t \cos u_0 + \theta_0). \end{aligned}$$

~~where c = k:~~

Thus

$$a(t) = \frac{b'(t)}{-\cot u_0} = -\tan u_0\, b'(t)$$

$$= +\tan u_0 \sin(t\cos u_0 + \theta_0)\cos u_0 \cdot C$$

$$\cancel{= c \sin u_0}$$

$$= +C\sin u_0 \sin(t\cos u_0 + \theta_0).$$

Now at $t=0$, we see the intial

$$V_0 = \cancel{c\cos(\theta_0)}\; C\sin u_0 \sin\theta_0\, X_u + C\cos\theta_0\, X_v.$$

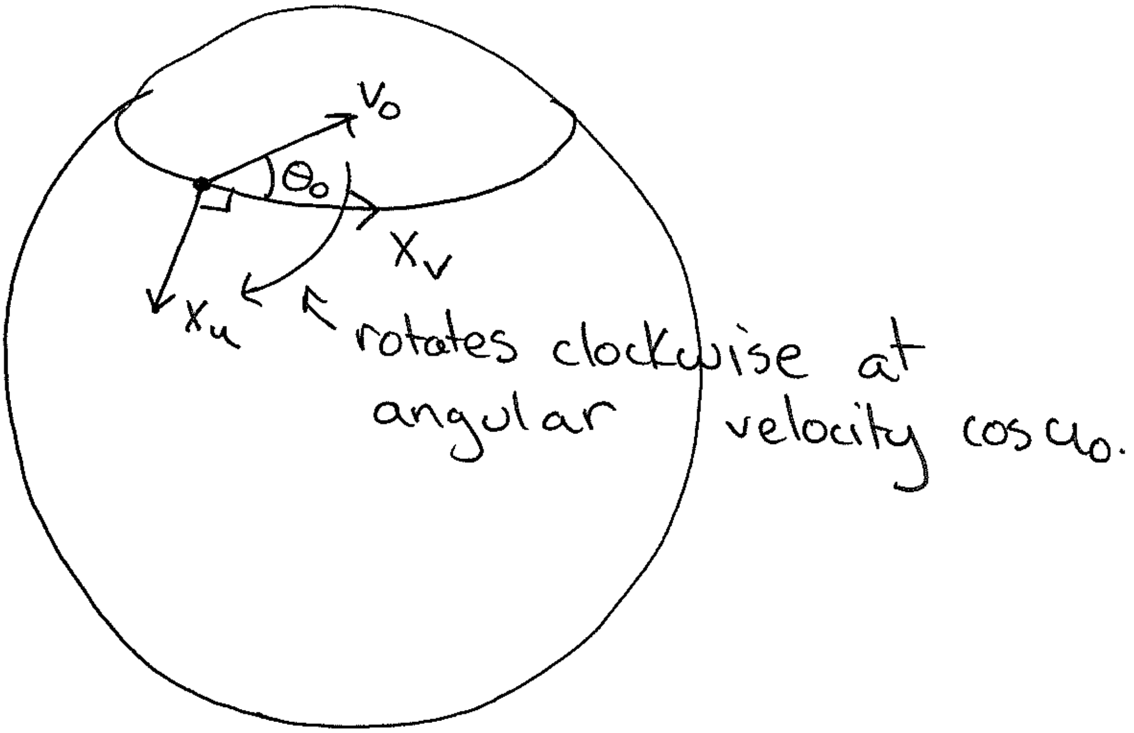
Since $F = \langle X_u, X_v\rangle = 0$, the X_u, X_v basis is orthogonal. But it's not orthonormal, $E = \langle X_u, X_u\rangle = 1$, but $G = \langle X_v, X_v\rangle = \sin^2 u_0$. So in the orthonormal basis $X_u, \frac{1}{\sin u_0} X_v$,

$$V_0 = (C\sin u_0 \sin\theta_0, C\sin u_0\cos\theta_0)$$

We conclude that

$$C \sin u_0 = |V_0|, \text{ or } C = \frac{|V_0|}{\sin u_0}$$

and that θ_0 is the angle V_0 makes with the X_u direction.



The intuition is that the covariant derivative $\nabla_{\gamma'(t)} \gamma'(t)$ points upwards on the sphere, so V has to spin in the opposite direction to stay parallel.

It's a cool fact that this explains the Foucalt pendulum – the forces on the pendulum as you rotate around the Earth are gravity (which is normal) and centripetal acceleration (which has a small tangential component), so the pendulum swing plane is parallel-transported around the Earth and rotates clockwise over time!