

# Parametrized curves, examples and constructions

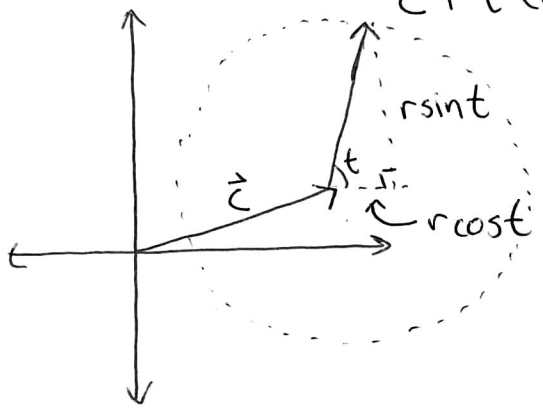
Recall that a parametrized curve is a map  $\vec{\alpha}: \mathbb{R} \rightarrow \mathbb{R}^2$ . We will now study some example curves.

Example. The circle of radius  $r$  with center  $\vec{c} = (c_1, c_2)$  in  $\mathbb{R}^2$  is described implicitly by

$$(x - c_1)^2 + (y - c_2)^2 = r^2$$

We can parametrize this curve by

$$\vec{c} + r(\cos t, \sin t) = \vec{\alpha}(t)$$



(2)

Notice that

$$\vec{\alpha}(t) = (c_1 + r \cos t, c_2 + r \sin t)$$

obeys

$$(\alpha_1(t) - c_1)^2 + (\alpha_2(t) - c_2)^2 =$$

$$= (r \cos t)^2 + (r \sin t)^2$$

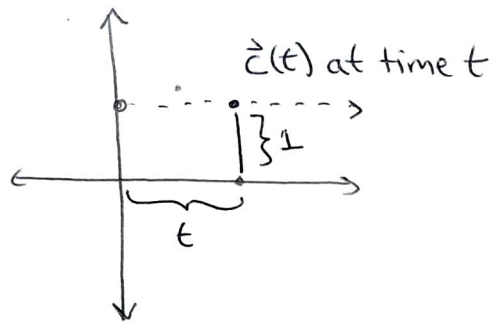
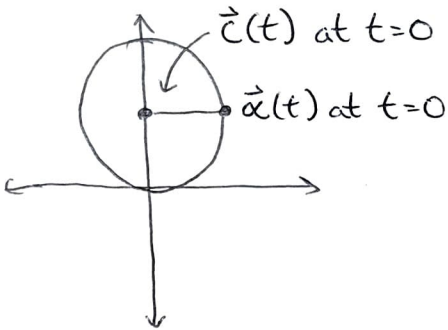
$$= r^2 (\cos^2 t + \sin^2 t) = r^2,$$

but there is more information in the ~~param~~ parametrization  $\vec{\alpha}(t)$  because it tells us when each point on the circle is reached.

Example 2.  $\vec{\alpha}(t) = (c_1 + r \cos(t^2), c_2 + r \sin(t^2))$  also parametrizes the circle of radius  $r$  and center  $\vec{c} = (c_1, c_2)$ .

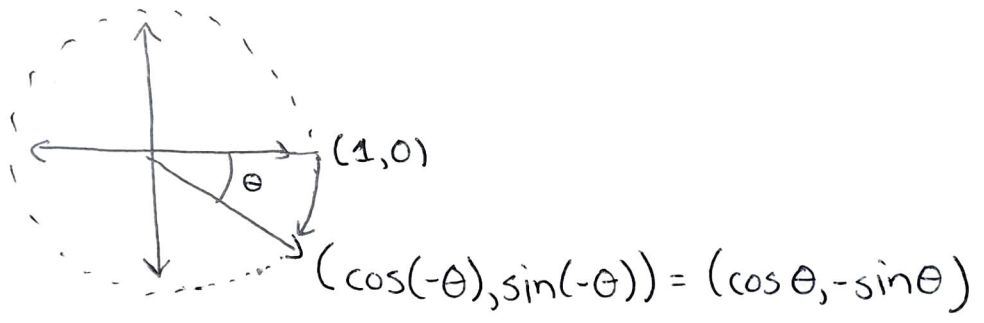
We can make some beautiful curves by combining sines and cosines.

Example. A unit circle starts with center at  $(0,1)$  and rolls along the pos. X axis. Parametrize the path of a point starting at  $(1,1)$ .



If the center of the circle is given by  $\vec{c}(t)$ , we can assume that the circle is rolling to the right at unit speed, so  $\vec{c}(t) = (t, 1)$ .

However, if ~~the~~ a unit circle has rolled  $t$  units forward, it has turned by an angle of  $t$  radians... in the clockwise direction.



This rotation carries the point at  $(1,0)$  ~~to the~~ (relative to the center) to the point at  $(\cos \theta, -\sin \theta)$  (relative to the center).

Adding these together:

$$\vec{x}(t) = (t + \cos t, 1 - \sin t)$$

We will work through a more elaborate <sup>⑤</sup>  
example of this type of motion  
in homework when we describe the  
square-wheeled car.

We often describe curves with a  
differential equation, so let's remember  
how to solve (easy) ODEs.

If  $u'(t) = F(u(t))$ , then

$$\frac{u'(t)}{F(u(t))} = 1$$

$$\int \frac{u'(t)}{F(u(t))} dt = \int 1 dt$$

$$\int \frac{1}{F(u)} du = \int 1 dt$$

integrate w.r.t.  $t$

$$du = u'(t) dt$$

so  $\int \frac{1}{F(u)} du = t + C.$

If we can do the integral on the left (6)  
to get some

$$G(u) = \int \frac{1}{F(u)} du$$

then we get an equation

$$G(u) = t$$

which we can try to solve for ~~u~~  $u(t)$ .

Example.  $u'(t) = u(t)^2$

$$\frac{u'(t)}{u(t)^2} = 1 \Rightarrow \int \frac{1}{u(t)^2} u'(t) dt = \int 1 dt$$

$$\Rightarrow \int \frac{1}{u^2} du = t + C \Rightarrow -\frac{1}{u} = t + C$$

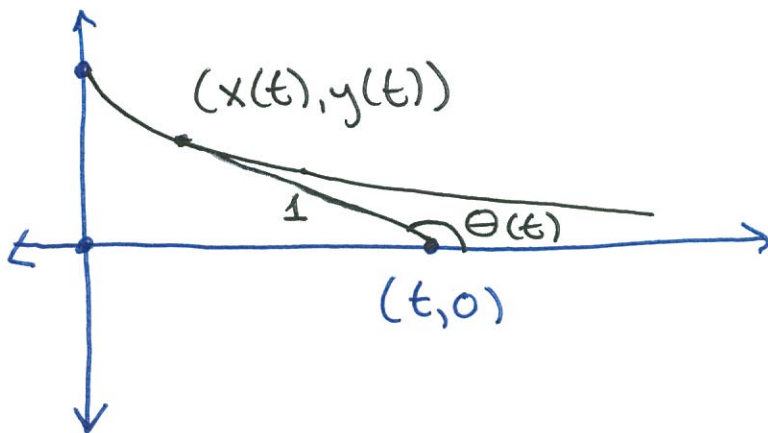
$$\Rightarrow u = -\frac{1}{t+C}$$

So  $u(t) = -\frac{1}{t+C}$ , and indeed  $u'(t) = \frac{1}{(t+C)^2} = u(t)^2$ .

①

The tractrix.

A mass is located at  $(0, 1)$  and pulled by a linkage of fixed length 1 moving along the x-axis at speed 1.



We know

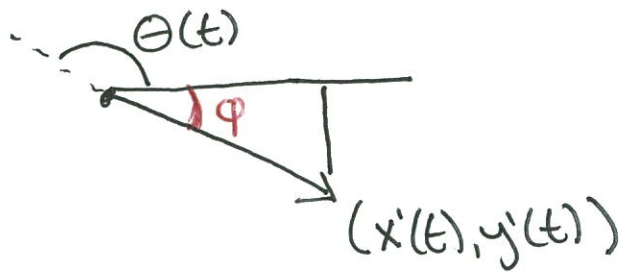
$$x(t) = t + \cos \theta$$

$$y(t) = \sin \theta$$

because of the length-1 constraint.

②

Less obviously, the linkage is tangent to the curve, so we know that we have a triangle



Since

$$\tan \phi = \frac{-y'(t)}{x'(t)}$$

← recall  $y'(t)$  is negative!

and  $\phi = \pi - \theta$ , the supplementary angle formula for tan tells us that

$$\tan \theta = \frac{y'(t)}{x'(t)} = \frac{\cos \theta \theta'(t)}{1 - \sin \theta \theta'(t)}$$

We can solve this formula for  $\theta'(t)$ .



③

$$\tan \theta (1 - \sin \theta \theta') = \cos \theta \theta'$$

$$\tan \theta - \tan \theta \sin \theta \theta' = \cos \theta \theta'$$

$$\tan \theta = \left( \cos \theta + \frac{\sin^2 \theta}{\cos \theta} \right) \theta'$$

→ multiplying through by  $\cos$ .

$$\sin \theta = \theta'$$

We can solve this by separation of variables:  $\sin \theta = \frac{d\theta}{dt}$ , so

$$\int \frac{1}{\sin \theta} d\theta = \int 1 dt$$

~~and~~ or

$$\int \csc \theta d\theta = -\ln(\csc \theta + \cot \theta) + C$$
$$= t$$

for some constant  $C$ .

at  $t=0$ , we have  $\Theta = \pi/2$ , so

(4)

$$\csc \pi/2 = 1, \quad \cot \pi/2 = \frac{0}{1} = 0$$

$$\text{and } -\ln(\csc \pi/2 + \cot \pi/2) = -\ln 1 = 0.$$

This means  $c=0$ . So

$$t = -\ln(\csc \Theta + \cot \Theta).$$

We want to solve this for  $\Theta$ .

Now

$$\csc \Theta + \cot \Theta = \frac{1 + \cos \Theta}{\sin \Theta}$$

We know

$$\cos^2 \frac{\Theta}{2} = \frac{1 + \cos \Theta}{2}$$

$$\sin^2 \Theta = 2 \cos^2 \frac{\Theta}{2} \sin^2 \frac{\Theta}{2}$$

(5)

so

$$\frac{1 + \cos \theta}{\sin \theta} = \frac{2 \cos^2 \theta/2}{2 \cos \theta/2 \sin \theta/2}$$

$$= \frac{\cot \theta/2}{\cancel{\cot} \theta/2}$$

(who said trig was useless!?) and

$$t = + \ln \tan \theta/2$$

we switched from cot to tan,  
killing the minus sign.

so the tractrix is parametrized  
by  $\theta$  if we substitute this back  
into

$$x(t) = t + \cos \theta$$

$$y(t) = \sin \theta$$

to get

$$x(\theta) = \cos \theta + \ln \tan \theta/2$$

$$y(\theta) = \sin \theta$$

Looking at start, end we see

$$\pi/2 \leq \theta < \pi$$

What about a  $t$  parametrization?

Well, exp-ing  $t = \ln \tan \theta/2$ , we get

$$\cancel{t} \approx e^t = \tan \theta/2$$

We now have to solve for  $\sin \theta$  and  $\cos \theta$  in terms of  $\tan \theta/2$ .

This is a trig exercise very similar to what we've already done.

⑥

⑦

Recall

$$\sin \theta = 2 \sin \theta/2 \cos \theta/2$$

$$\cos \theta = \frac{1 - \tan^2 \theta/2}{1 + \tan^2 \theta/2}$$

and that we can use the sin and cos in terms of tan formulas to write

$$\sin \theta/2 = \frac{\tan \theta/2}{\sqrt{1 + \tan^2 \theta/2}}$$

$$\cos \theta/2 = \frac{1}{\sqrt{1 + \tan^2 \theta/2}}$$

so we have

$$\sin \theta = \frac{2 \tan \theta/2}{1 + \tan^2 \theta/2}$$

Now plugging in  $e^t = \tan \theta/2$ ,

we get

⑧

$$\sin \theta = \frac{2e^t}{1+e^{2t}} = \frac{2}{e^{-t}+e^t} = \operatorname{sech} t$$

and

$$\cos \theta = \frac{1-e^{2t}}{1+e^{2t}} = \frac{e^{-t}-e^t}{e^{-t}+e^t} = -\operatorname{tanh} t$$

so we can also parametrize the tractrix by

$$(t - \operatorname{tanh} t, \operatorname{sech} t), t \geq 0.$$