# Comparing Graphs.

We now understand that the eigenvalues of a graph carry useful information. However, in general, it's hard to find formulas for them!

We now prove some theorems allowing to compare the spectrum of different graphs. We start by comparing matrices:

Definition. We write $A \succcurlyeq 0$ for a symmetric matrix $A$ if $A$ is positive semidefinite; (equivalently, all eigenvalues $\geq 0$ or $\langle \vec{v}, \vec{v} \rangle_A \geq 0$).

We write $A \succcurlyeq B$ if $A - B \succcurlyeq 0$.

Lemma. $A \succcurlyeq B \iff$ for all $\vec{v}$, $\langle \vec{v}, \vec{v} \rangle_A \geq \langle \vec{v}, \vec{v} \rangle_B$.

Proof. We just have to write out definitions.

$$\begin{aligned} \langle \vec{v}, \vec{v} \rangle_A \geq \langle \vec{v}, \vec{v} \rangle_B &\iff \langle \vec{v}, \vec{v} \rangle_A - \langle \vec{v}, \vec{v} \rangle_B \geq 0 \\ &\iff \langle \vec{v}, A\vec{v} \rangle - \langle \vec{v}, B\vec{v} \rangle \geq 0 \\ &\iff \langle \vec{v}, (A-B)\vec{v} \rangle \geq 0 \end{aligned}$$

This is true for all $\vec{v} \iff (A-B)$ is p.s.d. $\square$

Homework. If $A, B, C$ are symmetric $n \times n$ matrices, then

$$A \succcurlyeq B \text{ and } B \succcurlyeq C \implies A \succcurlyeq C$$

$$A \succcurlyeq B \implies A + C \succcurlyeq B + C \quad \text{(for any } C\text{!)}$$

Definition. If $G$ and $H$ are graphs, then we write $G \succcurlyeq H \iff L_G \succcurlyeq L_H$ where these are the Laplacian matrices of $G, H$.

Note. Not all graphs (or matrices) are comparable by $\succcurlyeq$, so this is a <u>partial</u> order on graphs, called <u>Loewner</u> <u>ordering</u>.

Lemma. If $G$ is a graph and $H$ is a subgraph of $G$, with same # of vertices then $L_G \succcurlyeq L_H$.

Proof. Recall that if $G$ is a <u>weighted</u> <u>graph</u>[1], with positive weights! then

$$\langle \vec{x}, \vec{x} \rangle_{L_G} = \sum_{\substack{v_i \sim v_j \\ \text{in } G}} \omega_{v_i \sim v_j} \left( \vec{x}(v_i) - \vec{x}(v_j) \right)^2$$

The edges of $H$ are a subset of the edges of $G$, so

$$\langle \vec{x}, \vec{x} \rangle_{L_H} = \sum_{\substack{v_i \sim v_j \\ \text{in } H}} \omega_{v_i \sim v_j} \left( \vec{x}(v_i) - \vec{x}(v_j) \right)^2$$

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1. We've only alluded to this before, but a weighted graph has scalar weights associated to vertices and edges.

sums a subset of the previous (non-negative) terms and so is smaller. $\square$

Definition. If $G$ is a weighted graph then $c \cdot G$ is the same graph with all weights multiplied by $c$.

Lemma. If $G \succcurlyeq c \cdot H$, then $\lambda_k(G) \geq c\lambda_k(H)$ for all eigenvalues $\lambda_k$ of $G$ and $H$.

Proof. Since $G \succcurlyeq c \cdot H$, $\langle \vec{v}, \vec{v} \rangle_{L_G} \geq \langle \vec{v}, \vec{v} \rangle_{cL_H} = c\langle \vec{v}, \vec{v} \rangle_{L_H}$ for all $\vec{v}$. Now by Courant-Fischer,

$$\lambda_k(G) = \min_{\substack{S \subseteq \mathbb{R}^V \\ \dim S = k}} \max_{\vec{x} \in S} \frac{\langle \vec{v}, \vec{v} \rangle_{L_G}}{\langle \vec{v}, \vec{v} \rangle}$$

$$\geq \min_{\substack{S \subseteq \mathbb{R}^V \\ \dim S = k}} \max_{\vec{x} \in S} \frac{c\langle \vec{v}, \vec{v} \rangle_{L_H}}{\langle \vec{v}, \vec{v} \rangle}$$

$$= c\,\lambda_k(H). \qquad \square$$

Putting our lemmas together:

Cor. If $G$ is a graph and $H$ is obtained by adding an edge or increasing the weight of an edge, for all $k$,

$$\lambda_k(G) \leq \lambda_k(H).$$

Definition. We say that $H$ is a $c$-approximation of $G$ if

$$cH \succcurlyeq G \succcurlyeq \frac{1}{c} H.$$

Later on, we'll construct very good

approximations of complete graphs ~~by~~ by random graphs!

We now give some examples:

Lemma. If $P_n$ is the path graph with edges $\{1 \leftrightarrow 2, 2 \leftrightarrow 3, \ldots, (n-1) \leftrightarrow n\}$ and $G_{1,n}$ has the single edge $1 \leftrightarrow n$, then

$$(n-1) P_n \succeq G_n.$$

Proof. We must show that for any $\vec{x} \in \mathbb{R}^n$ we have

$$(n-1) \langle \vec{x}, \vec{x} \rangle_{L_{P_n}} = (n-1) \sum_{i=1}^{n-1} (\vec{x}_{i+1} - \vec{x}_i)^2$$

$$\geq (\vec{x}_n - \vec{x}_1)^2 = \langle \vec{x}, \vec{x} \rangle_{L_{G_{1,n}}}.$$

So let $\Delta(i) = \vec{x}_{i+1} - \vec{x}_i$.

Since $\Delta(1) + \Delta(2) + \ldots + \Delta(n-1) = \vec{X}_n - \vec{X}_1$,

we must prove

$$☆ \quad (n-1) \sum_{i=1}^{n-1} \left(\Delta(i)\right)^2 \geq \left( \sum_{i=1}^{n-1} \Delta(i) \right)^2$$

Let's recall that for any vectors $\vec{u}, \vec{v}$ we know that

$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

$$\leq \|\vec{u}\| \, \|\vec{v}\|$$

so

$$\langle \vec{u}, \vec{v} \rangle^2 \leq \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle.$$

This is the Cauchy-Schwartz inequality.

Now if we let $\vec{u} = (\underbrace{1, \ldots, 1}_{n-1 \text{ times}})$ and $\vec{v} = (\Delta(1), \ldots, \Delta(n-1))$, we see that

$\star$ is exactly the statement

$$\langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle = (n-1) \sum \Delta(i)^2$$
$$\geq \langle \vec{u}, \vec{v} \rangle^2 = \left( \sum 1 \cdot \Delta(i) \right)^2. \quad \square$$

We will eventually use a fancier technique to show that for the path graph, $\lambda_2(P_n) \approx \pi^2/n^2$ for $n$ large. For now, we prove the (slightly) weaker inequality

Proposition. $\lambda_2(P_n) \geq \dfrac{6}{(n+1)(n-1)}$.

Proof. First, recall that if $K_n$ is the complete graph on $n$ vertices, then

$$L_{K_n} = \begin{bmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & n-1 \end{bmatrix} = nI - \mathbb{1}\mathbb{1}^T$$

The eigenvalues are $0$ (for multiples of $\vec{1}$) and $n$ (for all vectors $\perp$ to $\vec{1}$).

Thus $\lambda_2(K_n) = n$.

$$\text{Now} \quad L_{K_n} = \sum_{a<b} L_{G_{a,b}}$$

Laplacian of graph with 1 edge joining $a \leftrightarrow b$

Laplacian of graph with edges joining every $a \leftrightarrow b$.

For every $a<b$, let $P_{a,b}$ be the subgraph with edges $\{a \leftrightarrow a+1, a+1 \leftrightarrow a+2, \dots, \leftrightarrow b\}$ inside $P_n = \{1 \leftrightarrow 2, \dots, n-1 \leftrightarrow n\}$. Now

$$G_{a,b} \preccurlyeq (b-a)\, P_{a,b} \preccurlyeq (b-a)\, P_n$$

path graph lemma

subgraph lemma

Summing over edges,

$$K_n = \sum_{a<b} G_{a,b} \preccurlyeq \sum_{a<b} (b-a)\, P_n$$

but

$$\sum_{1 \le a < b \le n} (b-a) = \sum_{c=1}^{n-1} \underbrace{c}_{b-a=c}\,\underbrace{(n-c)}_{\text{occurs } n-c \text{ times}}$$

$$= n \sum_{c=1}^{n-1} c - \sum_{c=1}^{n-1} c^2$$

$$= n \, \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left(n - \frac{2n+1}{3}\right)$$

$$= \frac{n(n+1)}{2}\left(\frac{3n-2n-1}{3}\right)$$

$$= \frac{(n-1)n(n+1)}{6}.$$

So

$$L_{K_n} \preccurlyeq \frac{(n-1)n(n+1)}{6} L_{P_n}$$

and

$$n \leq \frac{(n-1)n(n+1)}{6} \lambda_2(P_n)$$

so

$$\lambda_2(P_n) \geq \frac{6}{(n-1)(n+1)}. \qquad \square$$

Now we consider the binary tree.

Definition. The complete binary tree $T_d$ of depth $(d+1)$ is the graph whose vertices are

$$\{ \ldots \text{all strings of 0's and 1's with length at most } d \text{, including the empty string} \ldots \}$$

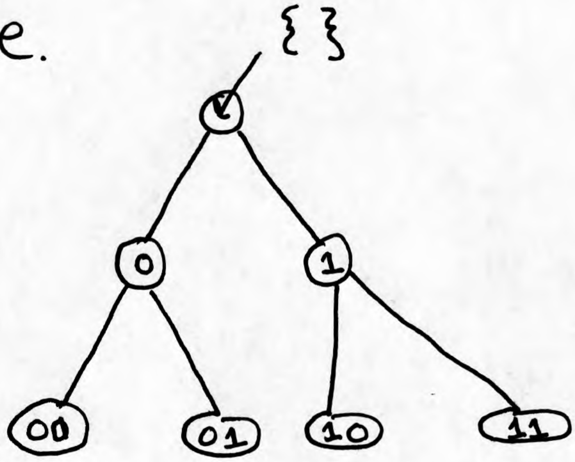
and whose edges are

$$\left\{ \begin{array}{l} b_1 b_2 \ldots b_k \longleftrightarrow b_1 b_2 \ldots b_k 0 \\ b_1 b_2 \ldots b_k \longleftrightarrow b_1 b_2 \ldots b_k 1 \end{array} \right\}$$

for all ~~k~~ $0 \leq k < d$.

This is the definition (implictly) most familiar to CS students.

Example.



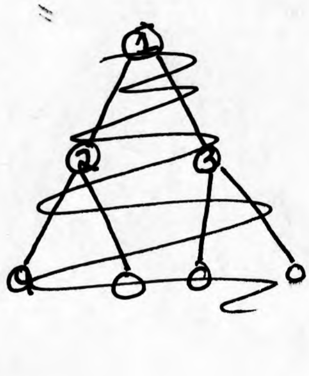

depth 3.

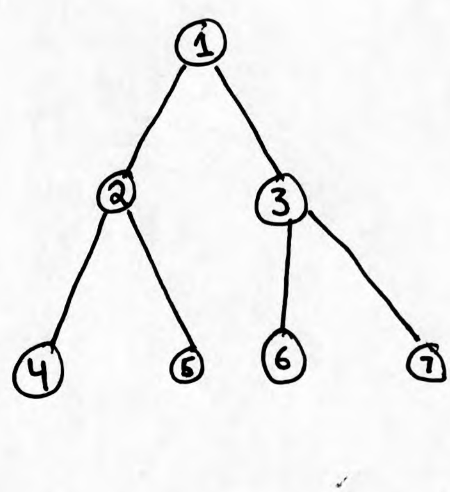
Here is a weirder definition (Lemma?)

Lemma. $T_d$ is isomorphic to the graph with vertices $\{1, \ldots, 2^{d+1} - 1\}$ with edges $\{ i \text{—} 2i,\ i \text{—} 2i+1 \}$ for $i < n$.

Proof. Homework.

Example.





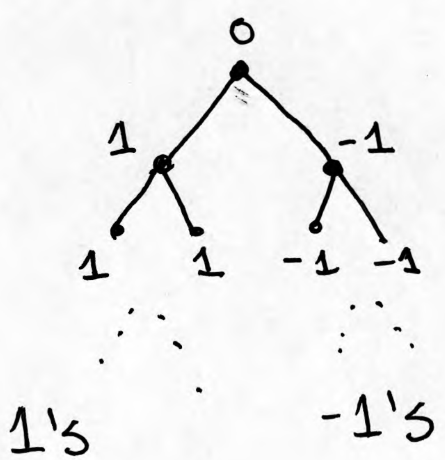
Proposition. $\dfrac{v}{\binom{v}{2} 2\log_2 v} \leq \lambda_2(T_d) \leq \dfrac{2}{2^{d+1}-2}$.

Proof. By the Courant-Fischer theorem, since $\lambda_1(L_{T_d}) = 0$ with eigenvector $\vec{1}$, we know

$$\lambda_2(L_{T_d}) = \min_{\langle \vec{x}, \vec{1} \rangle = 0} \frac{\langle \vec{x}, L_{T_d} \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle}$$

so we only need to construct an $\vec{x}$ with $\langle \vec{x}, \vec{1} \rangle = 0$ and $\langle \vec{x}, L_{T_d} \vec{x} \rangle / \langle \vec{x}, \vec{x} \rangle$ equal to $\frac{2}{2^{d+1}-2}$ to prove the rhs.

We let $\vec{x}(1) = 0$, $\vec{x}(2) = 1$, $\vec{x}(3) = -1$,




then extend down each half of the tree as shown.

There are $(2^d - 1)$ 1's, $(2^d - 1)$ -1's, and one zero in the $(2^{d+1} - 1)$-vector $\vec{x}$.

Clearly $\langle \vec{x}, \vec{1} \rangle = \sum_{i=1}^{2^{d+1}-1} \vec{x}_i = 0$. Further,

$$\lambda_2(T_d) \leq \frac{\langle \vec{x}, L_{T_d} \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle} = \frac{\sum_{a \leftrightarrow b} (\vec{x}(a) - \vec{x}(b))^2}{2(2^d - 1)}$$

Now $\vec{x}(a) = \vec{x}(b)$ for all $a \leftrightarrow b$ <u>except</u> $\vec{x}(2) - \vec{x}(1) = 1$, $\vec{x}(3) - \vec{x}(1) = -1$. So we have

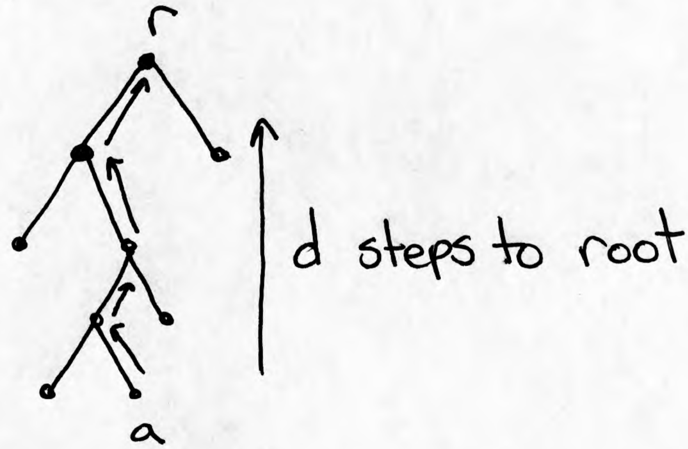
$$\sum_{a \leftrightarrow b} (\vec{x}(a) - \vec{x}(b))^2 = 2.$$

Combining these proves the upper bound.

Now we're going to prove the lower bound, again by comparing to the complete graph.

For each $a<b$, there is a unique path in $T_d$ between $a$ and $b$.

$T_d^{a,b}$




In a $(d+1)$-level tree, every vertex $a$ is no more than $d$ edges from the root vertex $r$, so $a$ and $b$ are no more than $2d$ edges apart.

Since $v = 2^{d+1} - 1$, we know

$$2^d < v$$

so

$$\log_2 2^d < \log_2 v$$

so

$$d < \log_2 v$$

and

$$2d < 2\log_2 v.$$

So

$$K_v = \sum_{a<b} G_{a,b} \preccurlyeq \sum_{a<b} (2d)\, T_d^{a,b} \quad \Big\downarrow \text{subgraph}$$

$$\downarrow <$$

$$\preccurlyeq \sum_{a<b} (2\log_2 v)\, T_d$$

$$= \binom{v}{2} 2\log_2 v \; T_d$$

so

$$\lambda_2(K_v) = v \leq \binom{v}{2} 2\log_2 v \; \lambda_2(T_d).$$

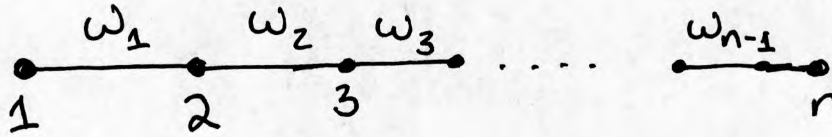
so

$$\lambda_2(T_d) \geq \frac{v}{\binom{v}{2} 2 \log_2 v}$$

But since $\binom{v}{2} = \frac{v(v-1)}{2}$, we can write

$$\geq \frac{1}{(v-1)\log_2 v} . \qquad \square$$

We're now going to extend our path graph inequality to one with weights.




Lemma. If $\omega_1, \dots, \omega_{n-1}$ are positive numbers, then

$$G_{1,n} \preccurlyeq \left( \sum_{i=1}^{n-1} \frac{1}{\omega_a} \right) \sum_{a=1}^{n-1} \omega_a \, G_{a,a+1}$$

called the <u>harmonic</u> mean of the $\omega_a$

Proof. Pick any $\vec{x} \in \mathbb{R}^n$ and let

$$\Delta(a) = \vec{x}(a+1) - \vec{x}(a)$$

as before. Now let $\gamma(a) = \Delta(a)\sqrt{\omega_a}$, and

$$\omega^{-1/2}(a) = \frac{1}{\sqrt{\omega_a}}.$$

We can compute

$$\sum_a \Delta(a) = \sum_a \Delta(a)\sqrt{\omega_a} \cdot \frac{1}{\sqrt{\omega_a}}$$

$$= \langle \vec{\gamma}, \vec{\omega}^{-1/2} \rangle.$$

Now $\langle \vec{\omega}^{-1/2}, \vec{\omega}^{-1/2} \rangle = \sum \frac{1}{\omega_a}$, while $\langle \vec{\gamma}, \vec{\gamma} \rangle = \sum \Delta(a)^2 \omega_a$. So we have

$$\left(\sum_a \Delta(a)\right)^2 \leq \left(\sum_a \frac{1}{\omega_a}\right)\left(\sum_a \Delta(a)^2 \omega_a\right).$$

Now as before

$$\langle \vec{x}, \vec{x} \rangle_{L_{G_{1,n}}} = \left(\vec{x}(n) - \vec{x}(1)\right)^2$$

$$= \left( \sum_{i=1}^{n-1} \Delta(a) \right)^2$$ ← because the sum of $\Delta(a)$ telescopes

$$\le \left( \sum_a \frac{1}{w_a} \right) \left( \sum_a \Delta(a)^2 w_a \right)$$ ← we just proved this on p. 18

$$= \cancel{\le} \left( \sum_a \frac{1}{w_a} \right) \sum_a w_a \left( \vec{x}(a+1) - \vec{x}(a) \right)^2$$

$$= \cancel{\le} \left( \sum_a \frac{1}{w_a} \right) \langle \vec{x}, \vec{x} \rangle_{\underbrace{\sum w_a L_{G_{a,a+1}}}_{\text{Laplacian of weighted path.}}} \quad \square$$

We will now be able to cleverly choose ~~more~~ weights to get more from this lemma.