Codazzi and Gauss Equations II

Things now start to get heavy:

$$S_p = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} = \frac{1}{EG-F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}$$

$$= \frac{1}{EG-F^2} \begin{bmatrix} \ell G - mF & mG - nF \\ mE - \ell F & nE - mF \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

we can use this to compute

$$n_u = D_{X_u} n = -S_p(\vec{X}_u) = \frac{1}{EG-F^2}\Big((\ell G - mF) X_u + (mE - \ell F) X_v\Big)$$

$$n_v = D_{X_v} n = -S_p(\vec{X}_v) = \frac{1}{EG-F^2}\Big((mG-nF) X_u + (nE - mF) X_v\Big).$$

We're in it to win it, so we return to this defining equations for the Christoffel symbols

$$X_{uu} = \Gamma^u_{uu} X_u + \Gamma^v_{uu} X_v + \ell \vec{n}$$

$$X_{uv} = \Gamma^u_{uv} X_u + \Gamma^v_{uv} X_v + m \vec{n}$$

and observe that

$$X_{uuv} = (\Gamma^u_{uu})_v X_u + \Gamma^u_{uu}\big(\overbrace{\Gamma^u_{uv} X_u + \Gamma^v_{uv} X_v + m\vec{n}}^{X_{uv}}\big)$$

$$+ (\Gamma^v_{uu})_v X_v + \Gamma^v_{uu}\big(\overbrace{\Gamma^u_{vv} X_u + \Gamma^v_{vv} X_v + n\vec{n}}^{X_{vv}}\big)$$

$$+ \ell_v \vec{n} + \ell\big(\underbrace{-c X_u - d X_v}_{\vec{n}_v}\big)$$

Collecting this in $X_u, X_v,$ and n, we have

$$= \left((\Gamma^u_{uu})_v + \Gamma^u_{uu}\Gamma^u_{uv} + \Gamma^v_{uu}\Gamma^u_{vv} - c\ell\right) X_u$$

$$+ \left((\Gamma^v_{uu})_v + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv} - d\ell\right) X_v$$

$$+ \left(\Gamma^u_{uu} m + \Gamma^v_{uu} n + \ell_v\right)\vec{n}$$

Now this is equal to X_{uvu}, which is

$$X_{uvu} = (\Gamma^u_{uv})_u X_u + \Gamma^u_{uv}\big(\overbrace{\Gamma^u_{uu} X_u + \Gamma^v_{uu} X_v + \ell\vec{n}}^{X_{uu}}\big)$$

$$+ (\Gamma^v_{uv})_u X_v + \Gamma^v_{uv}\big(\underbrace{\Gamma^u_{vu} X_u + \Gamma^v_{vu} X_v + m\vec{n}}_{X_{vu}}\big)$$

$$+ m_u \vec{n} + m\big(\underbrace{-a X_u - b X_v}_{n_u}\big)$$

As you'd expect, we collect these in $X_u, X_v, \vec{n}$:

$$X_{uvu} = \left((\Gamma^u_{uv})_u + \Gamma^u_{uv}\Gamma^u_{uu} + \Gamma^v_{uv}\Gamma^u_{uv} - am \right) X_u$$
$$+ \left((\Gamma^v_{uv})_u + \Gamma^u_{uv}\Gamma^v_{uu} + \Gamma^v_{uv}\Gamma^v_{uv} - bm \right) X_v$$
$$+ \left(\Gamma^u_{uv}\, l + \Gamma^v_{uv}\, m + m_u \right) \vec{n}$$

Equating the coefficients, we have three equations. We can ~~use~~ solve these to isolate the l, m, n and a, b, c, d variables:

$$lc - ma = (\Gamma^u_{uu})_v + \cancel{\Gamma^u_{uu}\Gamma^u_{uv}} + \Gamma^v_{uu}\Gamma^u_{vv}$$
$$- (\Gamma^u_{uv})_u - \cancel{\Gamma^u_{uv}\Gamma^u_{uu}} - \Gamma^v_{uv}\Gamma^u_{uv}$$

$$ld - mb = (\Gamma^v_{uu})_v + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv}$$
$$- (\Gamma^v_{uv})_u - \Gamma^u_{uv}\Gamma^v_{uu} - \Gamma^v_{uv}\Gamma^v_{uv}$$

$$l_v - m_u = \Gamma^u_{uv}\, l + (\Gamma^v_{uv} - \Gamma^u_{uu})\, m - \Gamma_{uu}\, n$$

Returning to our first computations,

$$\ell d - mb = \left(\ell(nE - mF) - m(mE - \ell F)\right)\frac{1}{EG-F^2}$$

$$= \left(\ell n E - \cancel{m\ell F} - m^2E + \cancel{m\ell F}\right)\frac{1}{EG-F^2}$$

$$= E\,\frac{\ell n - m^2}{EG-F^2}.$$

But we know

$$K = \det S_p = \det \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}$$

$$= \frac{\det \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}}{\det \begin{bmatrix} E & F \\ F & G \end{bmatrix}} = \frac{\ell n - m^2}{EG-F^2}.$$

Thus we have shown

$EK =$ < some mess of Christoffel symbols and their derivatives! >

This is the first <u>Gauss equation</u>.

Theorem. (Gauss's Theorema Egregium)
I_p determines Gauss curvature.

Proof. Already done - the Christoffel symbols determine K and I_p determines the Christoffel symbols. $\square$

Now you'll note that we have <u>three</u> more equations with ℓ, m, n and a, b, c, d and <u>two</u> equations with partials of ℓ, m, and n (from matching n components) if we also use $X_{uvv} = X_{vvu}$.

It turns out that the three give us equations for FK and GK:

$$FK = (\Gamma^u_{uv})_u - (\Gamma^u_{uu})_v + \Gamma^v_{uv}\Gamma^u_{uv} - \Gamma^v_{uu}\Gamma^u_{vv}$$

$$FK = (\Gamma^v_{uv})_v - (\Gamma^v_{vv})_u + \Gamma^u_{uv}\Gamma^v_{uv} - \Gamma^u_{vv}\Gamma^v_{uu}$$

$$GK = (\Gamma^u_{vv})_u - (\Gamma^u_{uv})_v + \Gamma^u_{vv}\Gamma^u_{uu} + \Gamma^v_{vv}\Gamma^u_{uv} - \Gamma^u_{uv}\Gamma^u_{uv} - \Gamma^v_{uv}\Gamma^u_{vv}$$

These are also called Gauss equations.

The remaining two are called Codazzi equations:

$$\ell_v - m_u = \ell\,\Gamma^u_{uv} + m(\Gamma^v_{uv} - \Gamma^u_{uu}) + n\,\Gamma^v_{uu}$$
$$m_v - n_u = \ell\,\Gamma^u_{vv} + m(\Gamma^v_{vv} - \Gamma^u_{uv}) + n\,\Gamma^v_{uv}$$

If we're really pressed, we can use the Gauss equations to compute Gauss curvature. But the main point was to show the Theorema Egregium.

Corollary. If M and M^* are locally isometric, their Gauss curvatures K and K^* are equal.

We say this means

"Gauss curvature is an isometry invariant".

Since this may be the first time

you've seen an invariant, let's unpack what this means.

$$M \text{ isometric to } M^* \Rightarrow K = K^*$$

is the same as

$$K \neq K^* \Rightarrow M \text{ not isometric to } M^*$$

This can be quite powerful.

Example. <u>No</u> map of the Earth can have correct angles, lengths and areas.

Proof. ~~For~~ The map is on a planar sheet of paper with $K=0$ and the curvature K^* of the sphere is 1. Hence I_p (map) is different from I_p^* (earth). But that means lengths and angles are different on map and Earth (if not, we could use

$$I_p(\vec{u},\vec{v}) = |\vec{u}||\vec{v}|\cos\theta = |\vec{u}|_*|\vec{v}|_*\cos\theta_* = I_p^*(\vec{u},\vec{v})$$

to show $I_p(\text{map}) = I_p^*(\text{earth})$.).

It is natural to ask whether

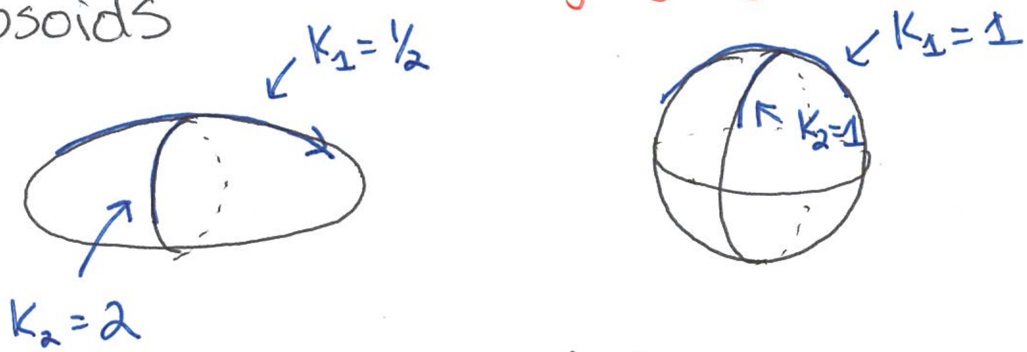
$$K = K^* \Rightarrow M \text{ isometric to } M^*$$

even though this more powerful statement does not follow from

$$M \text{ isometric to } M^* \Rightarrow K = K^*.$$

~~Example.~~

Answer: No. You'll do an example in homework, but the slightly bogus idea is that ellipsoids



are really not isometric.