The Codazzi and Gauss Equations

We are going to take a deep dive into notation and abstraction today, so we start with a big question:

Does Gauss curvature depend on how we embed M in space? Or do surfaces with matching I_p inner products have the same Gauss curvature always?

Examples. The cylinder and plane are (locally) isometric. For each, $\begin{bmatrix} E & F \\ F & G \end{bmatrix} = I$. Both have $K=0$, but they have different principal curvatures.

Conclusion. I_p doesn't determine II_p. But it might determine <u>det</u> II_p.

We start by considering the vectors X_{uu}, X_{uv} and X_{vv}. Since X_u, X_v and n are a basis for $\mathbb{R}^3$, there <u>are</u> functions so that

$$X_{uu} = \Gamma_{uu}^{u} X_u + \Gamma_{uu}^{v} X_v + l\vec{n}$$

$$X_{uv} = \Gamma_{uv}^{u} X_u + \Gamma_{uv}^{v} X_v + m\vec{n}$$

$$X_{vv} = \Gamma_{vv}^{u} X_u + \Gamma_{vv}^{v} X_v + n\vec{n}$$

We note that $X_{uv} = X_{vu}$, so $\Gamma_{uv}^{-} = \Gamma_{vu}^{-}$.

Example. Using our parametrization of the sphere as the surface of revolution with curve $\alpha(s) = (\overset{\sin s}{\cancel{\cos s}}, 0, \overset{\cos s}{\cancel{\sin s}})$,

$$X(u,v) = (\overset{\sin}{\cancel{\cos}}\, u \cos v, \cancel{\sin}\, \overset{\sin}{\cancel{\cos}}\, u \sin v, \overset{\cos}{\cancel{\sin}}\, u)$$

$$X_u = (\cos u \cos v, \cos u \sin v, -\sin u)$$

$$X_v = (-\sin u \sin v, \sin u \cos v, 0)$$

We recall that on the sphere

$$n = X(u,v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

For every surface of revolution $F = 0$, so the basis $\vec{X}_u, \vec{X}_v, \vec{n}$ is orthogonal (though not ortho<u>normal</u>). Thus

$$\langle X_{uu}, X_u \rangle = \Gamma^u_{uu} \langle X_u, X_u \rangle$$

$$\langle X_{uu}, X_v \rangle = \Gamma^v_{uu} \langle X_v, X_v \rangle$$

So we compute

$$\langle X_u, X_u \rangle = \cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u$$
$$= 1$$

$$\langle X_v, X_v \rangle = \sin^2 u \sin^2 v + \sin^2 u \cos^2 v$$
$$= \sin^2 u$$

and

$$X_{uu} = (-\overset{\sin}{\cancel{\cos}}\, u \cos v, -\sin u \sin v, -\cos u)$$

so

$$\langle X_{uu}, X_u \rangle = -\sin u \cos u \cos^2 v - \sin u \cos u \sin^2 v + \sin u \cos u = 0$$

$$\langle X_{uu}, X_v \rangle = \sin^2 u \cos v \sin v - \sin^2 u \cos v \sin v = 0.$$

and

$$\Gamma^u_{uu} = \Gamma^v_{uu} = 0.$$

To continue, we compute

$$X_{uv} = (-\cos u \sin v, \cos u \cos v, 0)$$

so

$$\langle X_{uv}, X_u \rangle = -\cos^2 u \cos v \sin v + \cos^2 u \cos v \sin v + 0 = 0.$$

$$\langle X_{uv}, X_v \rangle = \sin^2 v \cos u \sin u + \cos^2 v \sin u \cos u = \cos u \sin u$$

and

$$\Gamma^u_{uv} = 0 \quad \text{while} \quad \Gamma^v_{uv} = \frac{\cos u \sin u}{\sin^2 u} = \cot u$$

Finally, we have

$$X_{vv} = (-\sin u \cos v, -\sin u \sin v, 0)$$

and so

$$\langle X_{vv}, X_u \rangle = -\cos^2 v \sin u \cos u - \sin^2 v \sin u \cos u$$
$$= -\sin u \cos u$$

$$\langle X_{vv}, X_v \rangle = \sin^2 u \sin v \cos v - \sin^2 u \cos v \sin v$$
$$= 0$$

so we have

$$\Gamma^u_{vv} = \frac{-\sin u \cos u}{1} = -\sin u \cos u$$

$$\Gamma^v_{vv} = 0 \qquad \square$$

Let's stockpile those to check our work later. Here's another way to compute the Christoffel symbols:

symbols:

$$\begin{bmatrix} \Gamma_{uv}^{u} \\ \Gamma_{uv}^{v} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} E_v \\ \frac{1}{2} G_u \end{bmatrix}$$

$$\begin{bmatrix} \Gamma_{vv}^{u} \\ \Gamma_{vv}^{v} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{bmatrix}$$

These formulae can be checked for S^2.

Example. For the sphere,

$$E = 1, \quad F = 0, \quad G = \sin^2 u,$$

so

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \begin{bmatrix} \sin^2 u & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{\sin^2 u} = \begin{bmatrix} 1 & 0 \\ 0 & \csc^2 u \end{bmatrix}$$

Thus

$$\begin{bmatrix} \Gamma_{uu}^{u} \\ \Gamma_{uu}^{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \csc^2 u \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \Gamma_{uv}^{u} \\ \Gamma_{uv}^{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \csc^2 u \end{bmatrix} \begin{bmatrix} 0 \\ \sin u \cos u \end{bmatrix} = \begin{bmatrix} 0 \\ \cot u \end{bmatrix}$$

and

$$\begin{bmatrix} \Gamma_{vv}^{u} \\ \Gamma_{vv}^{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \csc^2 u \end{bmatrix} \begin{bmatrix} -\sin u \cos u \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin u \cos u \\ 0 \end{bmatrix}$$

So we <u>do</u> recover our previous compotation.

Proposition. I_p determines the Christoffel symbols.

This follows from our formulae above, which express the Γ_{ij}^{k} in terms of E, F, G and their derivatives.