Classification of points, Meusnier's formula

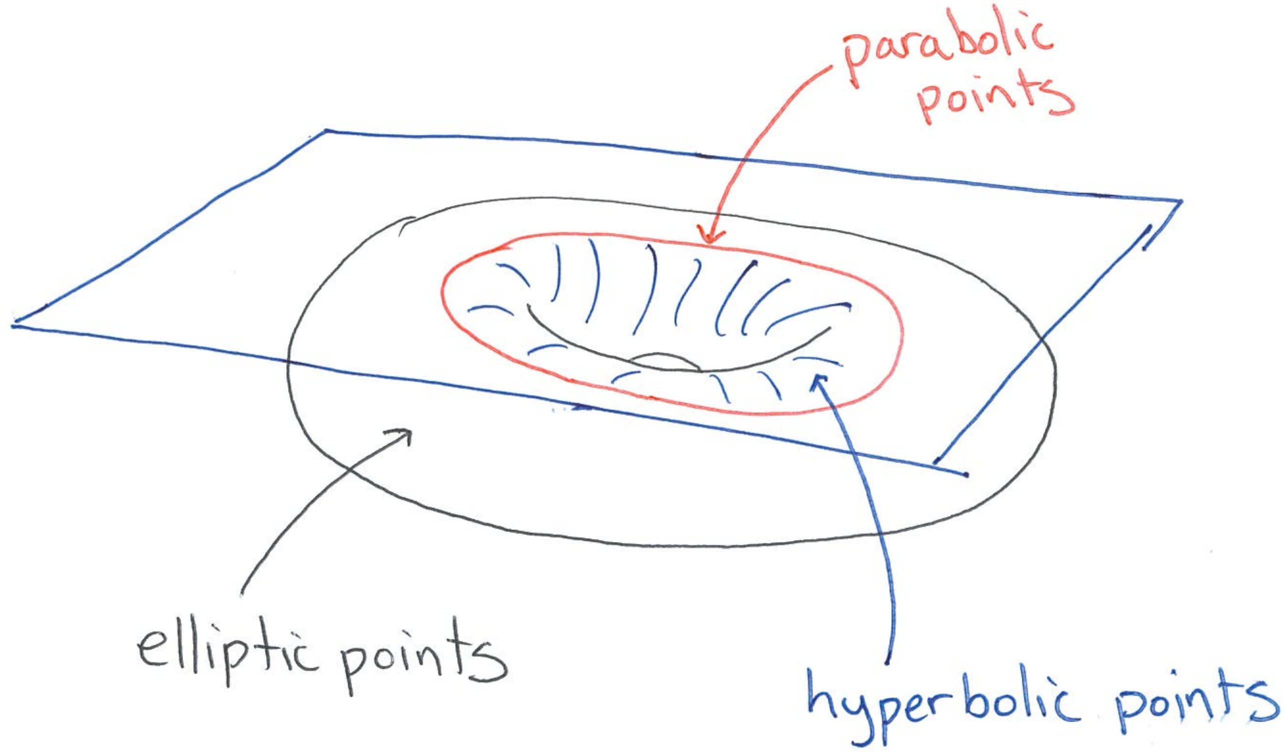
Last time, we revisted the classification of quadratics into ellipses, parabolae, and hyperbolae.

Defimitions. If

- $K = \det S_p > 0$, p is an *elliptic* point
- $K = \det S_p = 0$, one of κ_1, κ_2 is not zero, p is a *parabolic* point
- $K = \det S_p < 0$, p is a *hyperbolic* point

It's also useful to name some other cases.

- $\kappa_1 = \kappa_2$, p is an *umbilic* point
- $\kappa_1 = \kappa_2 = 0$, p is a *planar* point



Claim. On a torus of revolution, the top and bottom planar circles are composed of parabolic points.

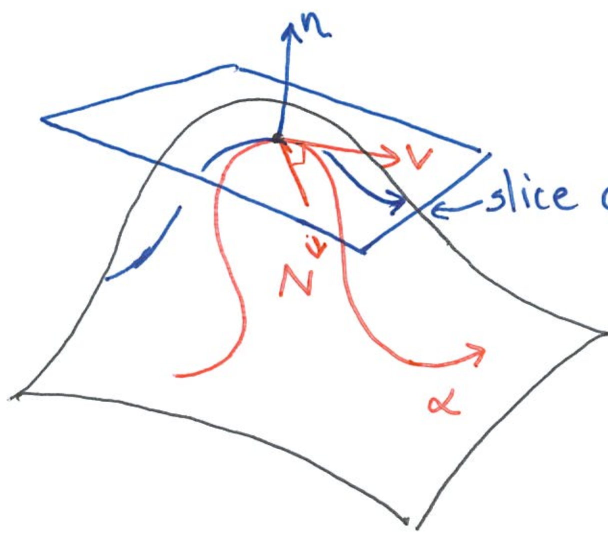
On a circle point, the normal does not change along the circle, so if V is tangent to the circle,

$$S_p(V) = D_V n = 0.$$

On the other hand, all other slice curves bend in the same direction,

so all slice curvatures are ≥ 0.
Since 0 is an extreme value for slice curvature, it is a principal curvature.

We now revisit slice curvatures.



Proposition. If α is any curve in M with unit tangent vector V at p, then

$$\mathrm{II}_p(v,v) = \langle \kappa N, \vec{n} \rangle = \kappa \cos \varphi$$

where φ is the angle between $\vec{n}$ and N.
We call this the <u>normal</u> curvature of α at p, denoted κ_n.

Proof. We've actually already done this computation, but to refresh.

$$\mathrm{II}_p(\vec{v},\vec{v}) = \langle -D_v\vec{n}, \vec{v}\rangle$$

$$= \langle -\vec{n}(\alpha(s))', T(s)\rangle$$

$\langle \vec{n}, T\rangle \equiv 0$ because α is in the surface

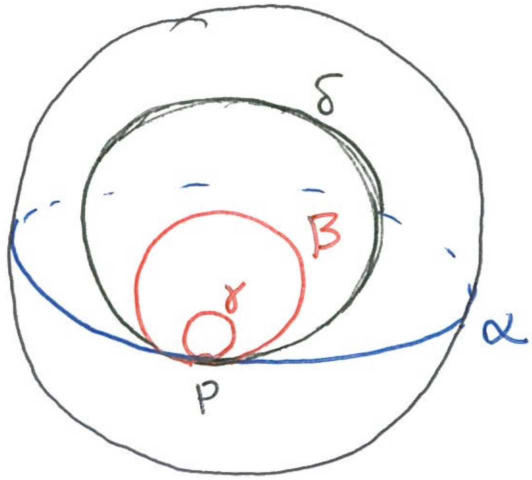
$$= \langle \vec{n}(\alpha(s)), T'(s)\rangle$$

$$= \langle \vec{n}, \kappa N\rangle. \quad \square.$$

☒ Examples.

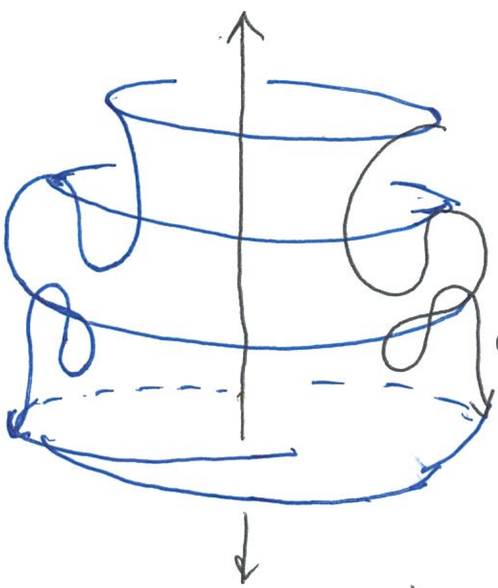
If α is an asymptotic curve, $\kappa_n \equiv 0$ along α, so $\langle N, \vec{n}\rangle \equiv 0$. This is true for all plane curves.

If two curves α, β in M share the same tangent vector at p, they have the same κ_n, even though their curvatures may be very different.



same normal curvature!

We now consider the general case of a surface of revolution.



$$X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

$$\alpha(u) = (f(u), \cancel{g(u)}\, 0, g(u))$$

We now run through the computations.

$$X_u = (f'(u)\cos v, f'(u)\sin v, g'(u))$$

$$X_v = (-f(u)\sin v, f(u)\cos v, 0)$$

$$n = \frac{(-g'(u)f(u)\cos v, -g'(u)f(u)\sin v, f'(u)f(u))}{\sqrt{f^2(u)^2 g'(u)^2 + f'(u)^2 f(u)^2}}$$

$$= \left(-g'(u)\cos v,\ -g'(u)\sin v,\ f'(u)\right)$$

since $f'(u)^2 + g'(u)^2 = 1$ (α arclength parametrized)

To continue,

$$X_{uu} = \left(f''(u)\cos v,\ f''(u)\sin v,\ g''(u)\right)$$

$$X_{uv} = \left(-f'(u)\sin v,\ f'(u)\cos v,\ 0\right)$$

$$X_{vv} = \left(-f(u)\cos v,\ -f(u)\sin v,\ 0\right)$$

So we have

$$E = \langle X_u, X_u\rangle = f'(u)^2\cos^2 v + f'(u)^2\sin v + g'(u)^2$$
$$= f'(u)^2 + g'(u)^2 = 1.$$

$$F = \langle X_u, X_v\rangle = -f'(u)f(u)\cos v\sin v + f'(u)f(u)\cos v\sin v$$
$$= 0.$$

$$G = \langle X_v, X_v\rangle = f(u)^2\sin^2 v + f(u)^2\cos^2 v = f(u)^2.$$

and

$$\ell = \langle \vec{n}, \vec{X}_{uu}\rangle = -g'(u)f''(u)\cos^2 v - g'(u)f''(u)\sin^2 v$$
$$+ f'(u)g''(u)$$
$$= f'(u)g''(u) - g'(u)f''(u)$$

$$m = \langle \vec{n}, \vec{X}_{uv} \rangle = f'(u)g'(u)\cos v \sin v$$
$$- g'(u)f'(u)\sin v \cos v + 0 = 0.$$

$$n = \langle \vec{n}, X_{vv} \rangle = g'(u)f(u)\cos^2 v + g'(u)f(u)\sin^2 v + 0$$
$$= f(u)g'(u).$$

Here we can write out the shape operator

$$S_p = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} = \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} \cdot \frac{1}{EG-F^2}$$

$$= \frac{1}{f(u)^2} \begin{bmatrix} f(u)^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f'(u)g''(u) - g'(u)f''(u) & 0 \\ 0 & f(u)g'(u) \end{bmatrix}$$

$$= \begin{bmatrix} f'(u)g''(u) - g'(u)f''(u) & 0 \\ 0 & \frac{g'(u)}{f(u)} \end{bmatrix}$$

This is already diagonalized, so we see

$$K_1 = f'(u)g''(u) - g'(u)f''(u) \qquad K_2 = g'(u)/f(u)$$

The principal directions are just X_u, X_v.

To simplify, we first observe

$$f'(u)^2 + g'(u)^2 \equiv 1, \quad \text{so}$$

$$2f'(u)f''(u) + 2g'(u)g''(u) = 0.$$

So

$$K = K_1 K_2 = \frac{1}{f(u)}\left[f'(u)\,g'(u)\,g''(u) - g'(u)^2 f''(u)\right]$$

$$= -\frac{1}{f(u)}\left[f'(u)^2 f''(u) + g'(u)^2 f''(u)\right]$$

$$= -\frac{f''(u)}{f(u)}.$$

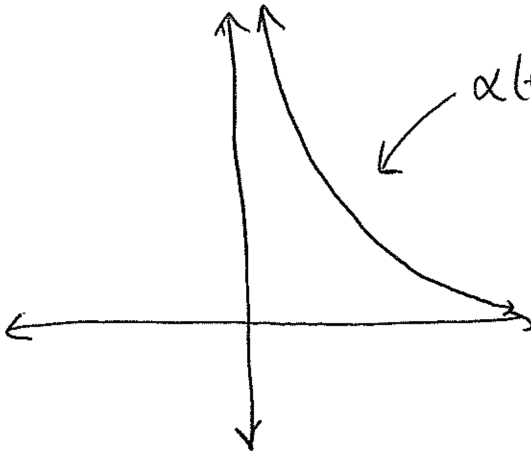
Example. The sphere of radius a is a surface of revolution with

$$\alpha(u) = \left(a\cos\left(\tfrac{u}{a}\right), 0, a\sin\left(\tfrac{u}{a}\right)\right).$$

so $f(u) = a\cos\left(\frac{u}{a}\right)$, $f'(u) = -\sin\left(\frac{u}{a}\right)$, and $f''(u) = -\frac{1}{a}\cos\left(\frac{u}{a}\right)$. Thus

$$K = -\frac{f''(u)}{f(u)} = \frac{1}{a}\,\frac{\cos(u/a)}{\cos(u/a)} = \frac{1}{a}.$$

Example. The surface of revolution of the tractrix is called the pseudosphere.



$\alpha(t) = (\cancel{\tanh t}, 0, \cancel{\operatorname{sech} t})$.

$\operatorname{sech} t \qquad t - \tanh t$

This is not an arclength parametrization, so we compute

$$\alpha'(t) = (1 - \operatorname{sech}^2 t, 0, \operatorname{sech}^2 t \tanh t)$$

$$|\alpha'(t)|^2 = 1 - 2\operatorname{sech}^2 t + \operatorname{sech}^4 t + \operatorname{sech}^2 t \tanh^2 t.$$

Now

$$\cosh^2 t - \sinh^2 t = 1,$$

so

$$1 - \tanh^2 t = \operatorname{sech}^2 t \quad \text{or} \quad 1 - \operatorname{sech}^2 t = \tanh^2 t$$

Writing everything in sech t,

$$|\alpha'(t)|^2 = 1 - 2\operatorname{sech}^2 t + \cancel{\operatorname{sech}^4 t} + \operatorname{sech}^2 t - \cancel{\operatorname{sech}^4 t}$$

$$= 1 - \operatorname{sech}^2 t = \tanh^2 t.$$

We now compute $K = -\dfrac{f''(s)}{f(s)}$

$$f(t) = \operatorname{sech} t$$

so

$$f'(s) = f'(t) \cdot \frac{dt}{ds} = f'(t) \Big/ \frac{ds}{dt} \quad \leftarrow |\alpha'(t)| = \tanh t$$

$$= \frac{f'(t)}{\tanh t} = \frac{\operatorname{sech} t \, \cancel{\tanh t}}{\cancel{\tanh t}}$$

miraculous cancellation!

$$= \operatorname{sech} t.$$

Of course, this means that

$$f''(s) = \operatorname{sech} t$$

by the same argument. Thus

$$K = -\frac{f''(s)}{f(s)} = -\frac{\cancel{\operatorname{sech}} t}{\cancel{\operatorname{sech}} t} = -1.$$

Example. The plane has constant normal vector so $K = K_1 K_2 = 0$.

We have now introduced 3 fundamental examples (we call these the <u>space forms</u>)

1) the sphere, $K \equiv 1$
2) the plane, $K \equiv 0$
3) the pseudosphere, $K = -1$.

We will return to these later on!