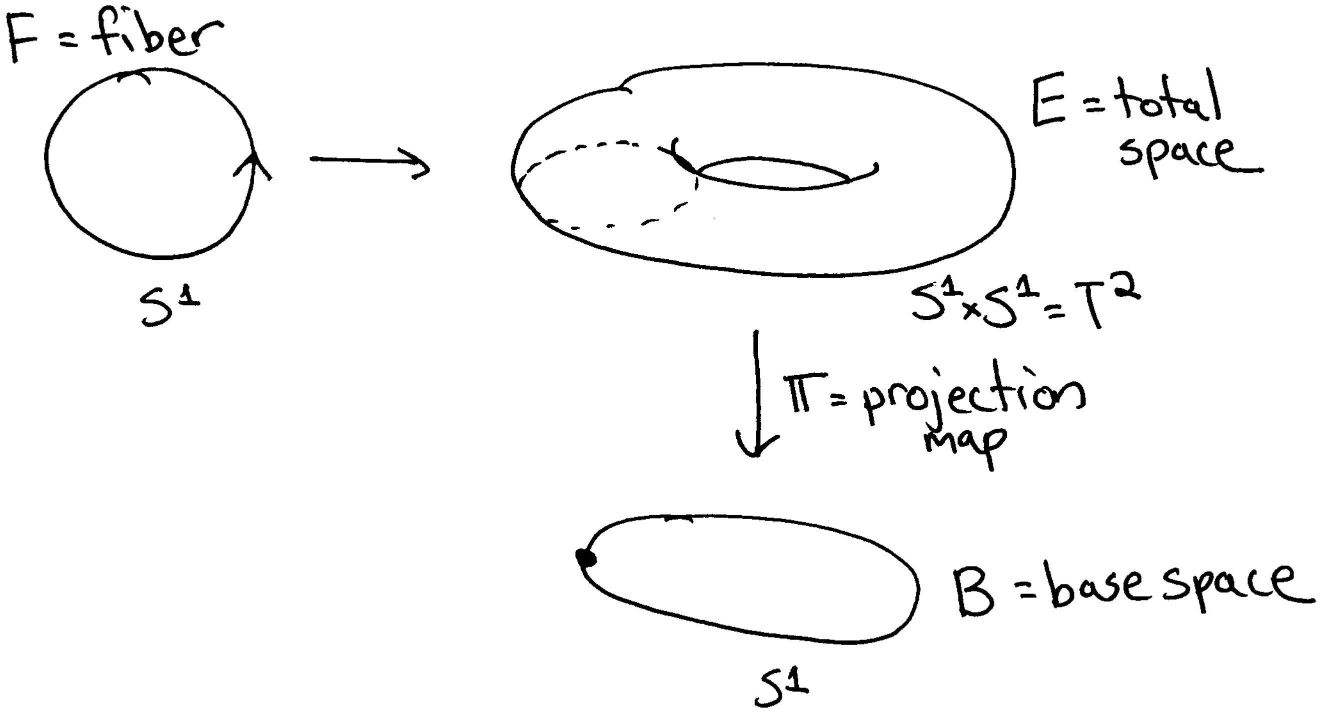
# Lecture 2. Vector Bundles

A <u>bundle</u> is best understood as a kind of "twisted product" of two manifolds.




A product space is one example, but a trivial one. Formally, we have

**Definition.** A real <u>vector bundle</u> $\xi$ (xi) over B consists of

1) a topological space E called the total space

2) a (continuous) map $\pi : E \to B$ called the projection map

3) for each $b \in B$ a vector space structure on $\pi^{-1}(b)$.

which obey the following conditions:

For each $b \in B$, $\exists$ a neighborhood $U$, an integer $n$, and a homeomorphism

$$h : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

so that for each $b \in U$, $x \mapsto h(b,x)$ is an isomorphism $\mathbb{R}^n \to \pi^{-1}(b)$.

We call $(U,h)$ a <u>local coordinate system for $\xi$ about $b$</u>.

If $U = B$, the bundle is trivial.

We call $\pi^{-1}(b)$ the <u>fiber over b</u>.

We usually assume that all fibers have the same dimension, $n$, making $\xi$ an <u>n-plane bundle</u>.

If everything is smooth and $\mathcal{U}$ is a <u>diffeomorphism</u>, then this is a <u>smooth vector bundle</u>.

Definition. Two vector bundles over the same space are <u>isomorphic</u> iff $\exists$ a homeomorphism $f: E(\xi_1) \to E(\xi_2)$ so that $F_b(\xi_1) \xrightarrow{f} F_b(\xi_2)$ is an isomorphism for all $b$.

Example. $B \times \mathbb{R}^n$ is the total space of the trivial bundle over $B$ if we let $\pi(b,x) = b$ and take the vector space structure

$$t_1(b,\vec{x}_1) + t_2(b,\vec{x}_2) = (b, t_1\vec{x}_1 + t_2\vec{x}_2)$$

on $\pi^{-1}(b)$.

Example. The tangent bundle, $TM$.

If the tangent bundle is trivial, we say $M$ is parallelizable.

Example. $T^2$ is parallelizable (prove it!)
$S^2$ is not parallelizable (by $\chi \neq 0$)

so we have our first example of a nontrivial vector bundle in $TS^2$.

Example. If $M \subset \mathbb{R}^n$, we can define the normal bundle of $M$ to be the subspace of $M \times \mathbb{R}^n$ of pairs

$$(x,v) \text{ s.t. } v \perp T_x M$$

This is usually denoted $\nu$.

Question. Is the normal bundle of $S^2$ trivial?

Question. Suppose you can embed $M$ in $\mathbb{R}^{n+1}$. Does that mean that $\nu$ is trivial?

Immerse?

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Example. Define $\mathbb{RP}^n$ as usual. The canonical line bundle over $\mathbb{RP}^n$ is the $\mathbb{R}$-bundle given by the subset of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ of $(x, v)$ so that $v$ is a multiple of $x$.

We call this bundle $\gamma_n^1$.

It is clear that $\gamma_n^1$ is locally trivial.

Theorem If $n > 1$, $\gamma_n^1$ is not trivial.

We will prove this using cross-sections.

Definition. A cross section $s$ of $\xi$ is a continuous function $s: B \to E$ which maps $b \mapsto F_b(\xi)$. The cross-section is <u>nowhere zero</u> if $s(b) \neq \vec{0}$ for all $b$.

Note that a <u>vector field</u> is a cross-section of the tangent bundle.

Proof. The trivial bundle has a nowhere zero cross-section. So suppose $s: \mathbb{R}P^n \to E(\gamma_n^1)$ is a cross-section. Consider

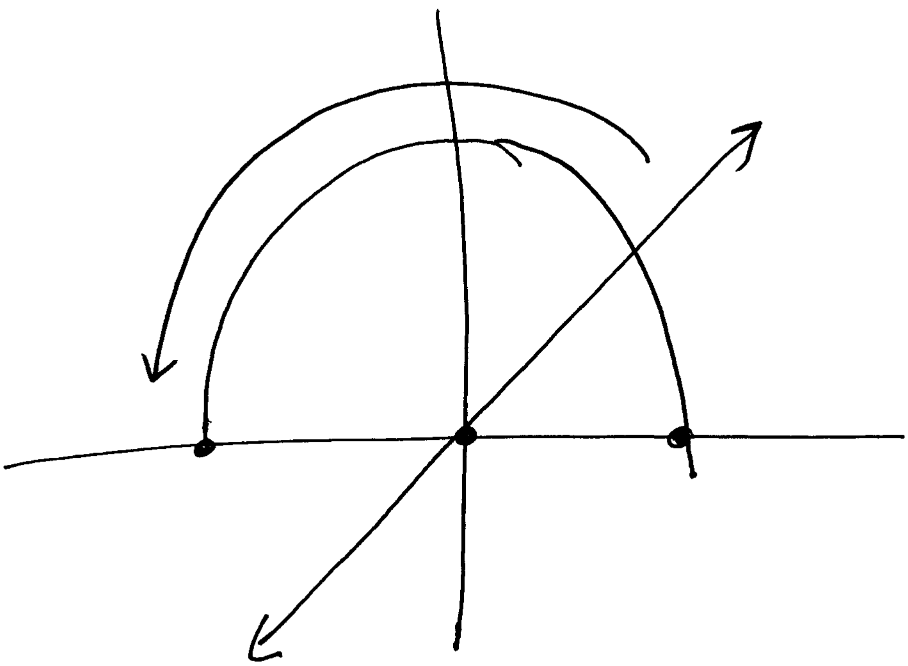
$$f: S^n \longrightarrow \mathbb{R}P^n \xrightarrow{s} E(\gamma_n^1).$$

Now $f(x) = (\{\pm x\}, t(x)\,x)$ where $t(x)$ is a continuous function of $x$. And

$$f(-x) = f(x), \quad \text{so} \quad t(-x) = -t(x).$$

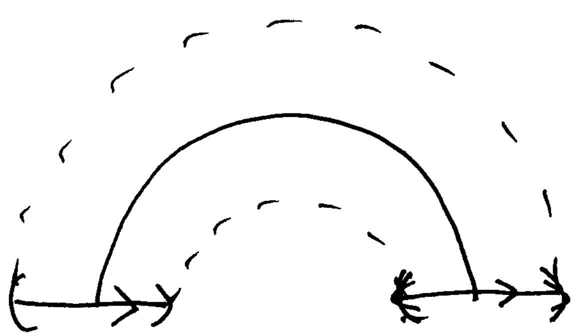
Now $S^n$ is connected, so $t(x_0) = 0$ for some $x_0$ on any path from $x$ to $-x$.

Explicit example. $\gamma_1^1$.



We identify $\mathbb{R}P^1$ with the upper ~~hemisph~~ half-circle (with endpoints identified).

Note that over this interval we have



an (open) strip, but the gluing reverses orientation on the ~~strip~~ boundary line.

Hence $\gamma_1^1$ = open mobius strip, while the trivial $S^1 \times \mathbb{R}^1$ bundle is the cylinder.

Now suppose we have a bunch of cross-sections $\{s_1, \dots, s_n\}$ of $\xi$.

Definition. A collection $\{s_1, \dots, s_n\}$ of cross sections of $\xi$ are nowhere-dependent if for each $b \in B$, $s_1(b), \dots, s_n(b)$ are linearly independent.

Lemma. Let $\xi, \eta$ be vector bundles over $B$. If $f: E(\xi) \to E(\eta)$ is continuous and maps each $F_b(\xi)$ isomorphically onto $F_b(\eta)$ then $f$ is a homeomorphism and $\xi \cong \eta$.

Proof. By assumption, $$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$ commutes, and $f$ is 1-1, onto and continuous. It remains only to show $f^{-1}$ is continuous, which follows from the fact that matrix inverses are cts.

**Theorem.** An $\mathbb{R}^n$-bundle $\xi$ is trivial $\Leftrightarrow$ it admits $n$ cross-sections which are nowhere dependent.

**Proof.** Easy consequence of Lemma.

Examples.

$S^1$ and $S^3$ are parallelizable.

## Euclidean vector bundles.

We can put a little more structure on our vector bundles by asking for an inner product (determined by a pos. def. quadratic form) on each vector space.

**Definition.** A Euclidean vector bundle is a real vector bundle together with a continuous function $\mu : E(\xi) \to \mathbb{R}$

which is quadratic and positive definite on each fiber.

This is called a <u>Euclidean metric</u> on $\xi$, and if $\xi = TM$, a <u>Riemannian metric</u> on $M$.

Note that $\mathbb{R}^n$ has the standard metric and the inclusion $M \hookrightarrow \mathbb{R}^N$ induces

$$TM \hookrightarrow T\mathbb{R}^N$$

which makes any $M \subset \mathbb{R}^N$ Riemannian.

<u>Lemma</u>. If $\xi$ is a trivial $n$-plane bundle and $\mu$ is a Euclidean metric on $\xi$, $\exists$ $n$ orthonormal cross-sections $s_1, \ldots, s_n$.

<u>Proof</u>. Continuity of Gram-Schmidt orthogonalization.

Problems. 2A-2B on p. 23