Smooth manifolds. Definitions.

We are going to define a bunch of diffeomorphism invariants of manifolds. Before we do that, we ought to review some definitions.

Definition. A function $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ is <u>smooth</u> (C^∞) if all partials exist and are continuous.

In general, let

$\mathbb{R}^A$ = the vector space of all functions from A to $\mathbb{R}$

if $x \in \mathbb{R}^A$, $\alpha \in A$, then the value of x on α is given by x_α, and called the α-coordinate of x.

If $f: Y \to \mathbb{R}^A$ is a function into $\mathbb{R}^A$, the α-coordinate of $f(y)$ is $f_\alpha(y)$.
We let $\mathbb{R}^A$ have the product topology (note: A can be infinite).

$f: Y \to \mathbb{R}^A$ is continuous $\iff$ $f_\alpha: Y \to \mathbb{R}$ cts for all α.

$(Y \subset \mathbb{R}^n)$ is smooth $\iff$ $f_\alpha: Y \to \mathbb{R}$ smooth for all α.

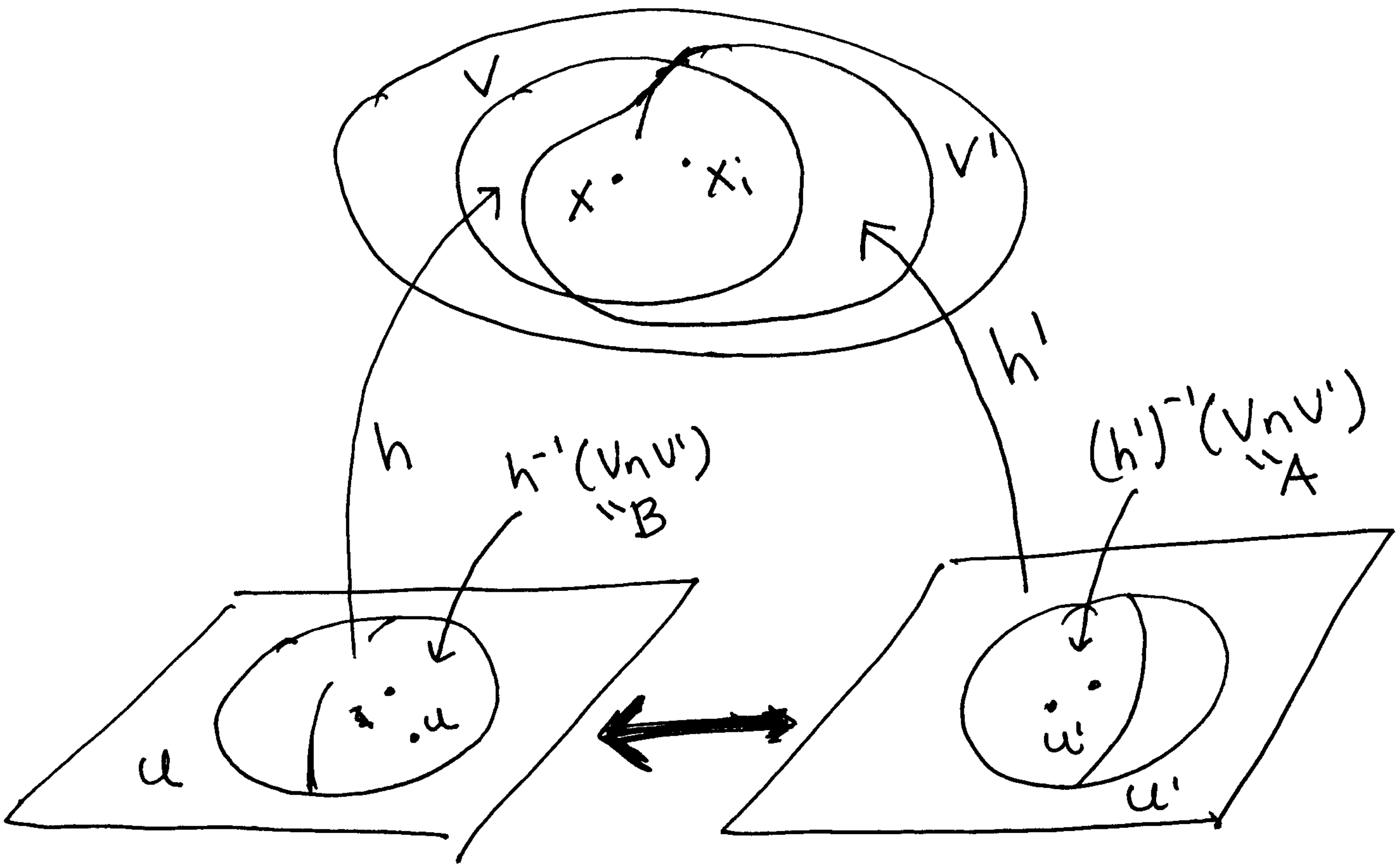
So in this case, we can define
$\partial f/\partial y_i$ as "the smooth function $Y \to \mathbb{R}^A$ so that the α-th coordinate is $\partial f_\alpha/\partial y_i$"

Definition. A subset $M \subset \mathbb{R}^A$ is a smooth manifold of dimension n (≥ 0) if for each $x \in M$ $\exists$ a smooth $h: U \subset \mathbb{R}^n \to \mathbb{R}^A$ with

a) $h(U) \subset M$ is a homeomorphic image of U containing x.

b) for each $u \in U$, $\left[\partial h_\alpha(u) / \partial u_j \right]$ has rank n.
(or the vectors $\partial h(u)/\partial u_1, \ldots, \partial h(u)/\partial u_n$ in the vector space $\mathbb{R}^A$ are linearly independent).

We call h a local parametrization for M at x. If we have



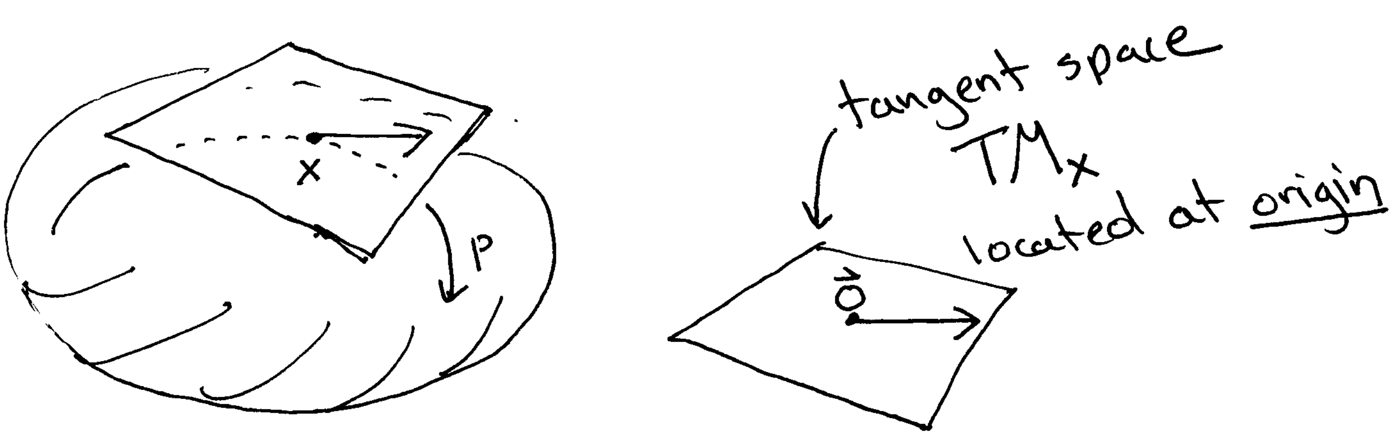
We can prove via inverse function theorem.

Lemma. In the setup above

$$u' \mapsto h^{-1}(h'(u')) : A \to B \text{ is smooth.}$$

Now we can define tangent vectors as expected.

Definition. A vector $\vec{v} \in \mathbb{R}^A$ is <u>tangent</u> to M at x if $\vec{v}$ is the velocity vector of some smooth path through x. The set of all such vectors is called the <u>tangent space</u> TM_x to M at x. (in your book, DM_x. Go figure.)



Lemma. TM_x is an n-dimensional real vector space if M is a smooth n-manifold.

Proof. A basis for TM_x is given by any local parametrization h as the set of partials of h at x in directions $e_1, \ldots, e_n$.

Definition. The tangent manifold of M is the subspace of $\mathbb{R}^A \times \mathbb{R}^A$ given by all (x, v) with $x \in M$, $v \in TM_x$.

Lemma. TM is a smooth manifold of dim $2n$.

Now what about equivalence of smooth mflds?

Definition. Suppose we have

$$f: M \subset \mathbb{R}^A \to N \subset \mathbb{R}^B,$$

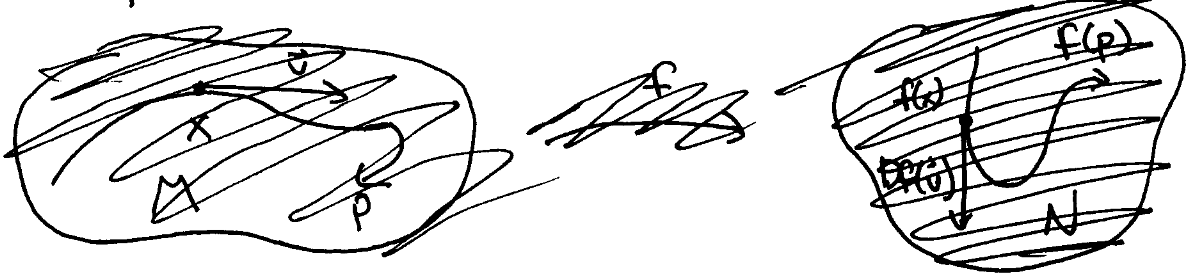
and a local parametrization h near $x \in M$. We say f is <u>smooth at x</u> if $f \circ h: U \to N \subset \mathbb{R}^B$ is smooth in a neighborhood of $h^{-1}(x)$, and f is <u>smooth</u> if it is smooth for all $x \in M$.

Definition. f is a diffeomorphism if f is one-one, onto, f and f^{-1} are smooth. ~~and the induced linear map~~

$$\cancel{Df_x : TM_x \to TN_x}$$

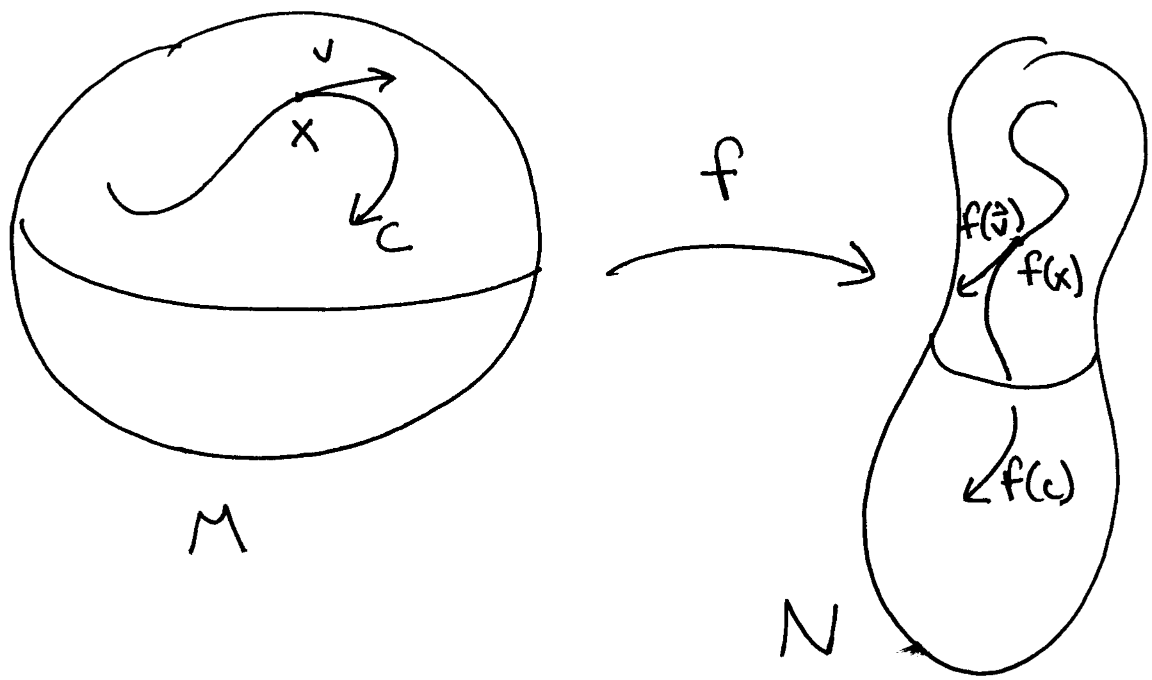
~~is nonsingular for all x.~~

~~we define~~ $\cancel{Df_x : TM_x \to TN_x}$ ~~by transforming paths and their velocity vectors~~



Note that if $M \xrightarrow{f} M' \xrightarrow{f'} M''$ are smooth, then $f' \circ f$ is smooth.

Now given a smooth map



we can map TM_x to $TN_{f(x)}$ by recalling that $TM_x = \{$ velocity vectors of smooth curves through $x \}$ and composing the curves with f.

We note that if $u_1, \ldots, u_n$ are coordinates on U, then Df_x can be written explicitly in terms of a local param. h,

$$Df_x\left(\sum c_i \frac{\partial h}{\partial u_i}\right) = \sum c_i \frac{\partial (f \circ h)}{\partial u_i}$$

We call Df_x the <u>derivative</u> or <u>Jacobian</u> of f. We have a smooth map

$$Df : TM \to TN$$

defined by $Df(x,v) = (f(x), Df_x(v))$.

Lemma. D has the following properties:

1) If M is smooth mfld, TM is smooth mfld.
2) If $f : M \to N$ smooth, $Df : TM \to TN$ smooth.
3) If $Id : M \to M$ is identity, $DId : TM \to TM$ is identity.
4) If we have $f \circ g$, then $(Df) \circ (Dg) = D(f \circ g)$.

Exercise. Show that $f : M \to N$ is a diffeomorphism $\Rightarrow$ $Df : TM \to TN$ is a diffeomorphism.

Consequence. We can study manifolds (up to diffeomorphism) by studying their tangent manifolds (up to diffeomorphism).

Now for something new (ish)!

Suppose we have $f : M \to \mathbb{R}$ a smooth function.

Then $Df_x : TM_x \to T\mathbb{R}_{f(x)} = \mathbb{R}$ is a linear map,

so $Df_x \in \underbrace{\mathrm{Hom}_{\mathbb{R}}}(TM_x, \mathbb{R})$.

$\mathbb{R}$-vector space homomorphisms $TM_x \to \mathbb{R}$

We call this the "total differential" of f at x, often written $df(x)$.

We think of this as the linear operator taking vectors to directional derivatives. Now

$$D(fg)_x = f(x)\, Dg_x + g(x)\, Df_x .$$

as operators.

Exercise. If the manifold is $\mathbb{R}^n$, $Df_x(v) = \langle \nabla f, v\rangle$. Use this to prove the above.

If we think of $C^\infty(M,\mathbb{R})$ as the vector space of smooth real-valued functions on ~~R~~ M, then holding $(x,v) \in TM$ fixed, we get a linear operator $X : C^\infty(M,\mathbb{R}) \to \mathbb{R}$

$$X(f) = Df_x(\vec{v})$$

Then

$$X(fg) = f(x)\, X(g) + X(f)\, g(x).$$

(slick! isn't it?)

A purely formal way (paging Dr. Hersonsky) to define TM would as the space of such linear operators on $C^\infty(M,\mathbb{R})$.

Important observation.

From this pov, we can't see a manifold apart from some particular embedding in a coordinate space $\mathbb{R}^A$.

Does "smoothness" depend on $\mathbb{R}^A$? On A?

Here's a (weird) canonical A for every smooth manifold M:

Let $F = C^\infty(M,\mathbb{R})$. Let

$$i : M \to \mathbb{R}^F \quad \text{be given by } i_f(x) = f(x).$$

Let M_1 = image of M under i in $\mathbb{R}^F$.

Lemma. M_1 is a smooth manifold in $\mathbb{R}^F$ and $i: M \to M_1$ is a diffeomorphism.

Proof. (obvious, but work it out together).

a) We must construct local parametrizations h around points in M_1. So pick $x_1 \in M_1$.

an actual point on M.

$$X_1 = i(x)$$

a vector of all values of smooth functions on M at x

We take the local param $h: U \to M$ and compose with i to get $h_1: U \to M_1$.

i) h_1 is smooth.

True if $(h_1)_f : U \to \mathbb{R}$ is smooth for every f. Well,

$$(h_1)_f(u) = i_f(h(u)) = f(h(u))$$

but f and h are smooth.

Lemma. M_1 is a smooth manifold in $\mathbb{R}^F$ and $i: M \to M_1$ is a diffeomorphism.

Proof (easy, but a good test on what we've done so far on this stuff).

1. i is onto.

   Obvious. M_1 is the image of i.

2. i is 1-1.

   Suppose $x \neq y$ in M. If f is a smooth function on M with $f(x) \neq f(y)$ then $i_f(x) \neq i_f(y)$. Hence $i(x) \neq i(y)$.

3. i is smooth.

   $i: M \to M_1$ is smooth $\iff$ $i_f: M \to \mathbb{R}$ is smooth for each $f \in F$. But $i_f(x) = f(x)$ and f is smooth by construction.

4. i^{-1} is continuous.

   ~~This is true if i is an open map~~

   ~~This is true if i is an open map. So consider~~
   ~~a small open neighborhood V of x∈M, and~~
   ~~map it forward under i_f to f(V) ∈ ℝ.~~

~~If~~

Take a sequence of ~~x~~ y_n in $M_1 \to y$ and consider $i^{-1}(y_n) = x_n$ in M. We wish to show that $x_n \to x = i^{-1}(y)$ in M.

Among the smooth functions on M are the coordinate functions $(h^{-1})_1, \ldots (h^{-1})_n$ of a local parametrization near x.

Since $(y_n)_{(h^{-1})_i} \to (y)_{(h^{-1})_i}$, we see that

$$h_i^{-1}(x_n) \to h_i^{-1}(i^{-1}(\cancel{x}y)) = h_i^{-1}(x).$$

or we have that the coordinates of the x_i approach those of x.

5. M_1 is a manifold.

Now we use the local parametrizations h of M to induce local parametrizations $i \circ h$ of M_i. We need only check that for a given $U \subset \mathbb{R}^n$,

$$\partial(i\circ h)/\partial u_1, \ldots, \partial(i\circ h)/\partial u_n$$

are linearly independent.

This is another exercise in tail-chasing, as we can consider the coordinates functions $(h^{-1})_1, \ldots, (h^{-1})_n$ of each vector and show that

Kronecker δ

$$\left(\partial(i\circ h)/\partial u_k\right)_{(h^{-1})_\ell} = \delta_{k\ell}$$

and so all these are clearly lin. indep.

6. i^{-1} is a smooth map.

We must take a local parametrization (U, V, h) at $x_1 \in M_1$ and show that each ~~coordinate~~ function $i^{-1}\circ h : U \to M \subset \mathbb{R}^A$ is smooth near $u_1 = h^{-1}(x_1)$.

But we know that

$$h = i \circ h', \text{ for some } \overset{\text{local param.}}{h'} : U \to M$$

so

$$i^{-1} \circ h = i^{-1} \circ i \circ h' = h',$$

which was smooth on M by construction.

7. i is a diffeomorphism.

Easy consequence of 1, 2, 3, and 6.

So that's a canonical coordinate space for M. Pretty cool! This suggests:

Definition. Let M be a set and F be a collection of functions on M which separate points. As before,

$$i: M \to M_1 \subset \mathbb{R}^F$$

is a bijection.

F is a <u>smoothness structure</u> on M if M_1 is a smooth manifold and F is the set of smooth functions on M.

Can I have different smoothness structures on the same* set? This is a deep question.

* homeomorphic

Important Exercises:

1-A, 1-B on p.11 in book.