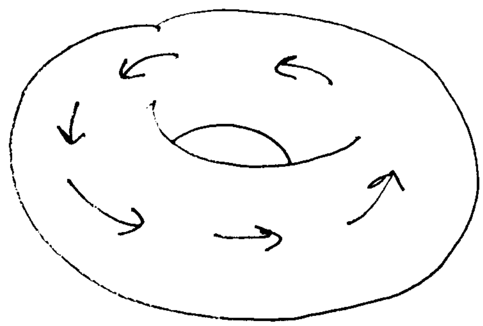
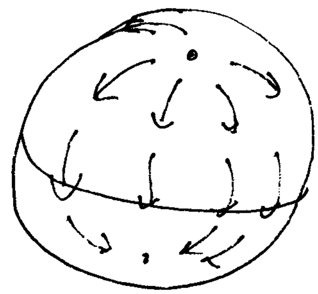
Characteristic Classes. Introduction

Suppose we have a smooth manifold M^n, and we want to know whether it admits a nonvanishing vector field.



T^2, yes



S^2, no

If we recall our differential topology, we remember that

$$\begin{array}{l}\#\text{ of zeros of a}\\ \text{vector field on } M\\ \text{(counted correctly)}\end{array} = \chi(M)$$

$$= \text{Euler characteristic.}$$

Now the Euler characteristic is so interesting because it ties together a lot of topological and geometric ideas about a manifold. We can write it as:

(1) If we decompose M^n as a CW-complex (a bunch of cells of different dimensions, glued together by attaching maps)

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \,(\#\text{ of cells of dimension } i)$$

so χ reflects the cell structure of M.

(2) If we calculate the homology of M^n, (or the cohomology) with real coefficients

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \,(\text{rank of } H_i(M))$$

so χ reflects the (co)homology of M

(3) If we remember that the <u>tangent bundle</u> TM is a smooth $2n$ manifold composed of the tangent spaces to M,

so

$p \in TM = (x, \vec{v})$ where $x \in M$ and $\vec{v} \in \mathbb{R}^n$ is a vector in the tangent space $T_x M$

and we take the submanifold $(x, 0)$ called the <u>0-section</u> of TM, then

$\chi(M)$ = intersection # of 0-section with itself in TM

so χ reflects <u>the structure of a canonical bundle over M.</u> and

χ is <u>an intersection number</u>.

(4) If we define the Gauss (or scalar) curvature of M by K, (a function on any particular embedding of M into some $\mathbb{R}^N$ which measures how fast the tangent space to M changes as we move around on M) then

$$\chi(M^n) = \frac{1}{\operatorname{vol} S^n} \int_M K(x)\, d\mathrm{vol}_M .$$

so

χ is <u>an integral of a curvature of M</u>.

(5) If you know the theory of DeRham cohomology, the fact that χ is an integral of an n-form over M which depends only on the homology of M might lead you to guess

$\chi(M)$ = evaluation of a class in $H^n(M)$ on the top class in $H_n(M)$.

So that

χ is <u>a cohomology class on M</u>.

This last is something that we'll prove in the course (you should not expect to have known it already).

So is the Euler characteristic the end of the story? (Well, no, or this would be a very short class!) Our work is based on 4 mathematicians:

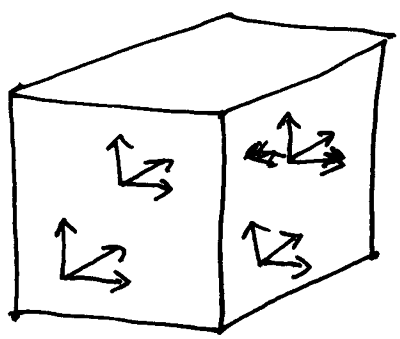
Stiefel Whitney Pontrjagin Chern

Idea: Suppose you want to generalize Euler class (χ) to consider sets of linearly independant vector fields on manifolds?

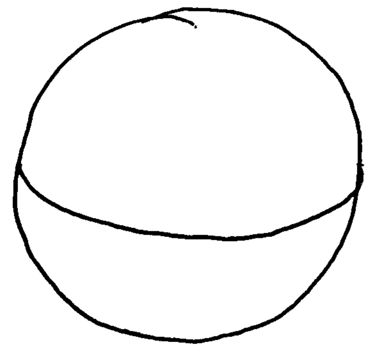
Then $\exists$ other classes which count the number of such vector fields.

Theorem. (Stiefel) Every compact orientable 3 manifold has a set of 3 nowhere vanishing vector fields which are linearly independent at every point of the mfld.

Examples.



$T^3 = S^1 \times S^1 \times S^1$



S^3 = unit quaternions so these are a Lie group.

We can take any ~~field~~ basis for tangent space at 1 and push it around by multiplication.

Exercise It isn't hard to construct the 3 fields on S^3 explicitly. The first one is $(-y, x, -w, z)$. What are the others?

Harder. Can you play this game on S^7?
Exercise

These classes then allow us to get new and interesting information about manifolds. They will be

1. Stiefel-Whitney classes. w_i

   Associated to any "vector bundle" B (or to the tangent bundle TM of a smooth manifold).

   $w_i \in H^i(TM, \mathbb{Z}/2\mathbb{Z})$, $w_i = 0$ for $i > n$.

Here is an amazing application for these.

Pontrjagin/Thom theorem.

A smooth compact manifold M^n is the boundary of a smooth compact manifold B^{n+1} $\iff$ all Stiefel-Whitney classes of M^n vanish.

We will prove this later, but note that it implies $\mathbb{RP}^3$ is the boundary of a 4 manifold. Which one?

Another amazing result, proved with these

Theorem. $\mathbb{RP}^{(2^k)}$ cannot be smoothly embedded in $\mathbb{R}^{(2^k)-1}$.

For example, $\mathbb{RP}^2$ can be <u>immersed</u> in $\mathbb{R}^3$, but not embedded.

We will then go on to define

2. Chern classes.

Associated to any <u>complex</u> vector bundle, B.

$$c_i \in H^{2i}(B, \mathbb{Z})$$

and

3. Pontrjagin classes.

Built from Chern classes, they live in

$$p_i \in H^{4i}(B, \mathbb{Z}).$$

These classes can be defined for triangulated (and even topological) manifolds as well as smooth ones.

As such, they are key to distinguishing the 4 types of topological equivalence for manifolds. ~~X~~ X and Y.

a) homotopic (or homotopy-equivalent)

$\exists$ $f: X \to Y$ and $g: Y \to X$ so that

$f \circ g: Y \to Y$ is homotopic to Id_Y and

$g \circ f: X \to X$ is homotopic to Id_X.

b) homeomorphic

$\exists$ $f: X \to Y$, $f^{-1}: Y \to X$ so that f is 1-1, onto and f, f^{-1} are continuous.

c) pl-equivalent (or combinatorially equivalent)

X and Y are triangulated and $\exists$ a subdivision of each triangulation X' and Y' so that $\exists$ a ~~f~~ homeomorphism X', Y' which is linear on each triangle.

~~d) diffeomorphic~~

d) diffeomorphic

X and Y are smooth, ∃ a smooth map $f: X \to Y$ so that f is 1-1, onto and Df and Df^{-1} are always nonsingular.

Key Question of 1960's topology.

If two manifolds X and Y are Known to be homotopic (when) are they
homeomorphic
pl-equivalent

actually homeomorphic ?
pl-equivalent
diffeomorphic

They are the key to: the "exotic 7-spheres":

[Milnor] There exists a triangulated 7-manifold which is homotopic, homeomorphic, and pl-equivalent to S^7, but not diffeomorphic to S^7.

This theorem will (hopefully!) be the climax of our course.

Our practical aim is to make you very comfortable with cohomology and with vector bundles by the end of the semester.

These are fundamental tools which will serve you well in the rest of your mathematical life.