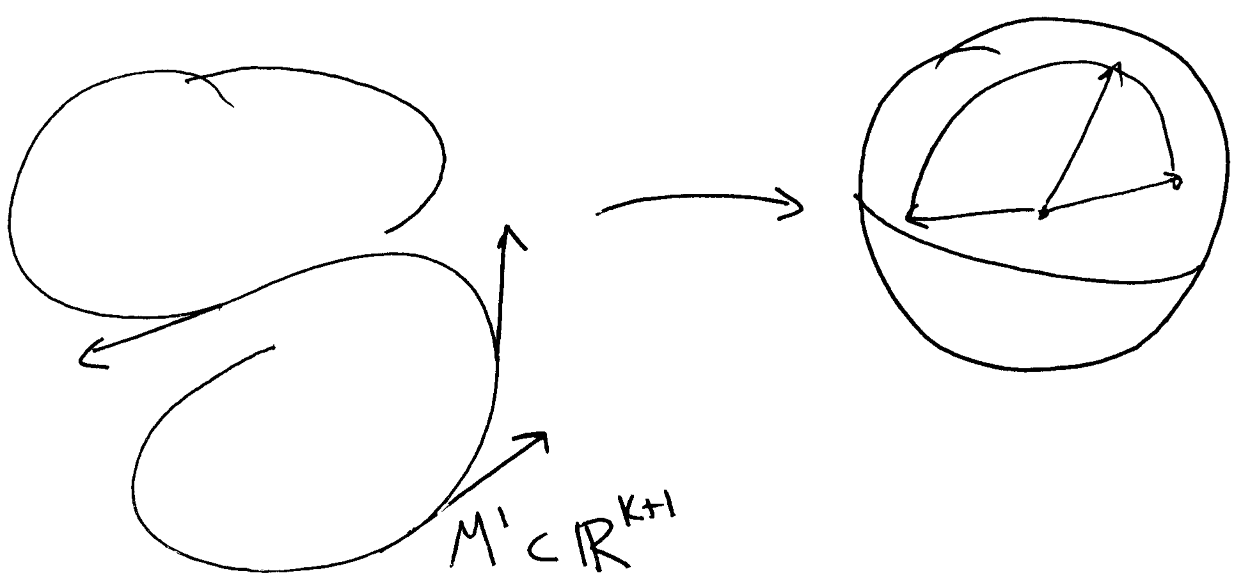
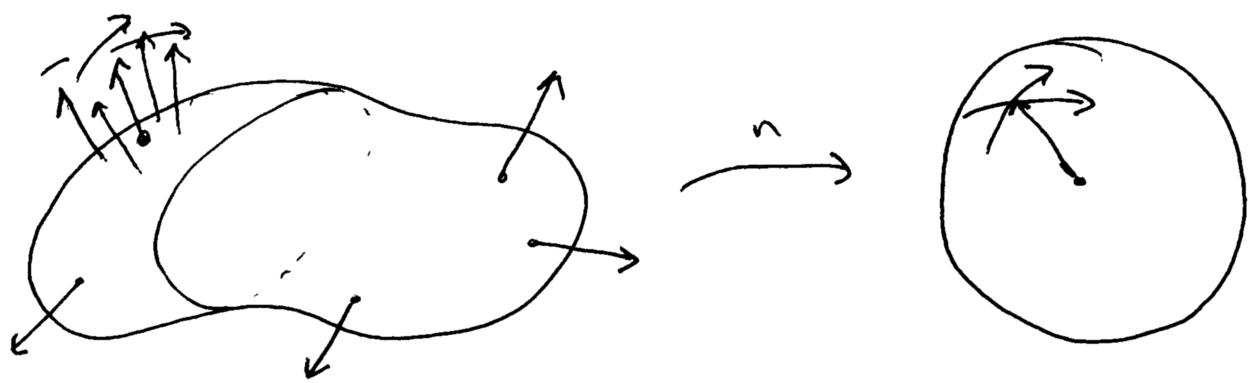
# Grassmannians and Universal Bundles.

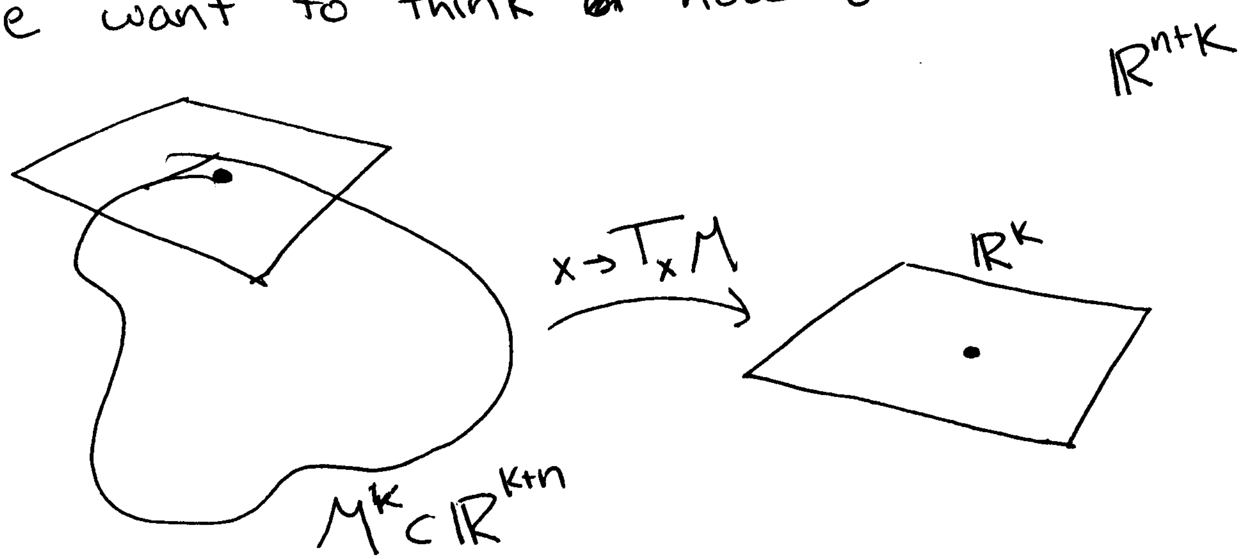
Our goal now is to present a new way to think about vector bundles over a manifold $M$. Our idea is to show that every bundle $\xi$ maps into a "universal" vector bundle. Here's some motivation for how this works.




Given $M^1 \subset \mathbb{R}^{k+1}$, we can construct the tangent indicatrix curve in $S^n$, by taking all unit tangents. Similarly, we defined curvature for submanifolds $M^k \subset \mathbb{R}^{k+1}$ in terms of the ~~vol~~ volume in $S^k$ of the image of the normal vector $n$.




In general, we need an orientation to define these maps. But clearly there is a map $M^1 \to \gamma^1_{k+1}$ given by $x \to T_x M$. We want to think ~~of~~ now of this map



as a map into a new space.

<u>Definition.</u> The <u>Grassmann manifold</u> $G_n(\mathbb{R}^{n+k})$ is the set of $n$-planes through $\vec{0}$ in $\mathbb{R}^{n+k}$.

<u>Definition.</u> A $n$-<u>frame</u> in $\mathbb{R}^{n+k}$ is an $n$-tuple of linearly independent vectors in $\mathbb{R}^{n+k}$.

<u>Definition.</u> The <u>Stiefel manifold</u> $V_n(\mathbb{R}^{n+k})$ is the set of $n$-frames in $\mathbb{R}^{n+k}$. (also <u>frame manifold</u>)

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Now we must say something about the topology on these sets.

a) $V_n(\mathbb{R}^{n+k}) \cong$ $n \times (n+k)$ matrices $A$ of full rank

$\cong$ $n \times (n+k)$ matrices $A$ s.t. $AA^T$ has full rank

$AA^T$ ← Gram matrix

$\cong$ inverse image of $\mathbb{R} - \{0\}$ under $A \mapsto \det(AA^T)$

$\cong$ an open subset of $\mathbb{R}^{n(n+k)}$

This topologizes $V_n(\mathbb{R}^{n+k})$

b) There is a natural map given by "span".

$$V_n(\mathbb{R}^{n+k}) \xrightarrow{\pi} G_n(\mathbb{R}^{n+k})$$

c) We give $G_n(\mathbb{R}^{n+k})$ the quotient topology induced by this map.

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We also can construct

$$V_n^0(\mathbb{R}^{n+k}) = \underline{\text{orthogonal}}\ n \text{ frames}$$

and use it to construct (the same) topology on $G_n(\mathbb{R}^{n+k})$.

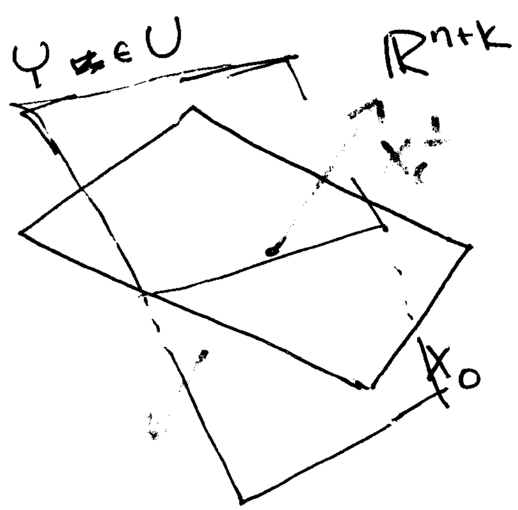
Lemma. The Grassmannian $G_n(\mathbb{R}^{n+k})$ is a compact topological manifold of dimension $nk$.

The map $X \mapsto X^\perp$ from $G_n(\mathbb{R}^{n+k}) \to G_k(\mathbb{R}^{n+k})$ is a homeomorphism.

<u>Proof</u>

We cut to the chase and prove only that each point in $G_n$ has a neighborhood homeomorphic to $\mathbb{R}^{nk}$.

Pick $X_0 \cong \mathbb{R}^n \subset \mathbb{R}^{n+k}$. We can write




$$\mathbb{R}^{n+k} = X_0 \oplus X_0^\perp$$

Let $U \subset G_n$ consist of $n$-planes $Y$ s.t. orthogonal projection

$\pi : \mathbb{R}^{n+k} \to X_0$

maps $\Upsilon$ <u>onto</u> $X_0$. Now then each vector $\vec{x} \in \mathbb{R}^{n+k}$ can be written $(\vec{v}, \vec{w})$ with $\vec{v} \in X_0$, $\vec{w} \in X_0^\perp$. Now the n-plane

$$\Upsilon = \{ (\vec{v}, \vec{w}) \mid (\vec{v}, \vec{w}) \in \Upsilon \}$$

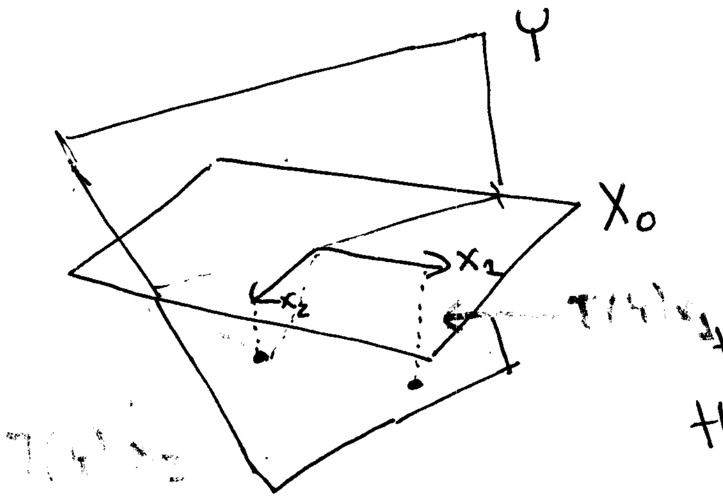
is the graph of a linear map from $X_0$ to $X_0^\perp$. (After all, it's linear by the fact $\Upsilon$ is a subspace, well-defined because $\pi : \Upsilon \to X_0$ is an isomorphism.)

Thus we have a map

$$T : U \to \mathrm{Hom}(X_0, X_0^\perp) \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k) \cong \mathbb{R}^{nk}.$$

<u>Claim.</u> $T$ is a homeomorphism.

It is clear that $T$ is 1-1.




To show $T$ is onto, let $T(Y) \in \mathrm{Hom}(X_0, X_0^\perp)$ and we can define $Y$ to be the image of the linear map

$$x \mapsto x + T(Y)x$$

from $X_0$ to $X_0 \oplus X_0^\perp = \mathbb{R}^{n+k}$.

It is easy to see $T, T^{-1}$ are continuous.

$\square$

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We note that there were other things to check ($G_n$ is compact, Hausdorff) but they are easy.

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The second part comes down to checking that $X^\perp$ depends continuously on $X$, which is an exercise in chasing definitions.

We now build a bundle $\gamma^n(\mathbb{R}^{n+k})$.

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \left\{ (X,\vec{x}) \;\middle|\; \begin{array}{l} X \subset \mathbb{R}^{n+k} \text{ is an } n\text{-plane and} \\ \vec{x} \in \mathbb{R}^{n+k} \text{ is in } X \end{array} \right\} \\ & & \Big\downarrow \pi \\ & & G_n(\mathbb{R}^{n+k}) \end{array}$$

The projection map carries $(X,\vec{x})$ to $X$.

Lemma. This bundle is locally trivial.

Proof. As before, given $X_0 \in G_n(\mathbb{R}^{n+k})$, let $U$ be set of $Y \in G_n(\mathbb{R}^{n+k})$ so that orthogonal projection $p_Y: Y \to X_0$ is onto. $p_Y$ is also 1-1, so we can define

$$h: U \times X_0 \to \pi^{-1}(U)$$

by $h(Y,x) = (Y, p_Y^{-1}(x)) = (Y, x + T(Y)x)$.
This is continuous (because $T$ was) and $h^{-1}(Y,y) = (Y, p_Y(y))$ is continuous because $p_Y$ is. $\square$

Now we define:

<u>Definition</u>: The (generalized) Gauss map

$$\bar{g}: M \to G_n(\mathbb{R}^{n+k})$$

is the function which takes $x \mapsto T_xM$. It is the base map of a bundle map

$$g: TM \to \gamma^n(\mathbb{R}^{n+k})$$

which takes $(x, v) \mapsto (T_xM, v)$.

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Lemma. For <u>any</u> n-plane bundle $\xi$ over a compact $B$ there exists a bundle map into $\gamma^n(\mathbb{R}^{n+k})$ for large enough $K$.

Claim. To construct a bundle map $f: \xi \to \gamma^n(\mathbb{R}^m)$ it suffices to build (ctsly)

$$\hat{f}: E(\xi) \to \mathbb{R}^m$$

which is linear and injective on each fiber of $\xi$. Then we let, for $e \in E(\xi)$

$$f(e) = \left(\hat{f}(\text{entire fiber } F_{\pi(e)}(\xi)), \hat{f}(e)\right).$$

Proof. Since the space $E(\gamma^n(\mathbb{R}^{n+k})) \subset G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$, it is clear that

$$f: E(\xi) \to E(\gamma^n(\mathbb{R}^{n+k})).$$

by linearity of $\hat{f}$ on fibers of $\xi$.

It is clearly continuous in the $\mathbb{R}^{n+k}$ variables.

To see that this is continuous as a map from $E(\xi) \to G_n(\mathbb{R}^{n+k})$, we just ~~check~~ observe at $e \in E(\xi)$ that

$$e \in h(U \times \mathbb{R}^n), \quad U \subset B(\xi)$$

where $h$ is a homeomorphism. So since $\hat{f}$ is continuous from $E(\xi)$ to $\mathbb{R}^m$,

$$h \circ \hat{f} : U \times \mathbb{R}^n \to \mathbb{R}^m$$

is linear on each $\mathbb{R}^n$, and 1-1. Choosing a frame on $\mathbb{R}^n$, say $x_1, \ldots, x_n$, we get a cts map

$$U \times \mathbb{R}^n \longrightarrow V_n(\mathbb{R}^m)$$

so that

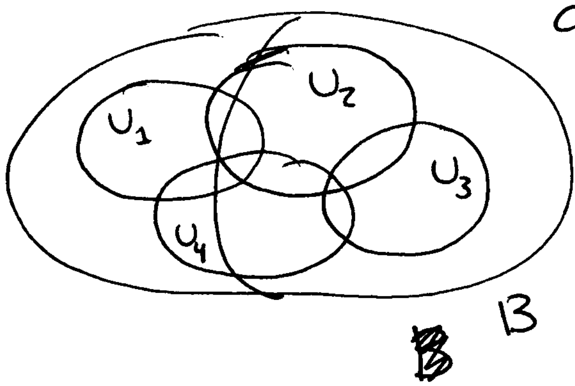
$$\begin{tikzcd} & V_n(\mathbb{R}^m) \arrow[d, "\pi"] \\ U \times \mathbb{R}^n \arrow[ur, bend left] \arrow[r, "\hat{f}\,(\text{fiber})"'] & G_n(\mathbb{R}^m) \end{tikzcd}$$

commutes. Thus $\hat{f}$ (fiber) is continuous as well. $\square$

Proof (of lemma).

We need to build a map which is linear and injective on each fiber of $\xi$ to some $\mathbb{R}^m$.

Cute idea. Locally trivialize in a collection of sets $U_1, \dots, U_r$ covering $B$ and let $m = nr$.




(A separate $\mathbb{R}^n$ for every $U_i$!)

Pick some $V_i$ covering $B$ with $\overline{V_i} \subset U_i$ and $W_i$ covering $B$ with $\overline{W_i} \subset V_i$. Now take partitions of unity $\lambda_i$ so that $\lambda_i$ cts and

$$\lambda_i = \begin{cases} 1, & \text{inside } W_i \\ 0, & \text{outside } V_i \\ (0,1), & \text{in between} \end{cases}$$

Now $\xi|_{U_i}$ is always trivial, so there is a map

$$h_i : \pi^{-1}(U_i) \longrightarrow \mathbb{R}^n$$

which is an isomorphism on each fiber over $U_i$. Now let

$$h_i' : E(\xi) \to \mathbb{R}^n$$

be given by $h_i'(e) = \lambda_i(e)\, h_i(e)$, and define

$$\hat{f} : E(\xi) \to \underbrace{\mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n}_{r \text{ copies}}$$

by

$$\hat{f}(e) = (h_1'(e), \ldots, h_r'(e)). \quad \square$$

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Now this worked fine for <u>compact</u> manifolds, where we had a finite cover of trivial neighborhoods.

But what if the base is <u>not</u> compact?

<u>Definition.</u> Let $\mathbb{R}^\infty = \mathbb{R} \times \dots \times \mathbb{R} \times \dots$ with the <u>product</u> topology (that is, an element of $\mathbb{R}^\infty$ is a vector with a <u>finite</u> # of nonzero elts).

<u>Definition</u>. The infinite Grassmannian

$$G_n = G_n(\mathbb{R}^\infty)$$

is the set of $n$-dimensional linear subspaces of $\mathbb{R}^\infty$, with the direct limit topology from

$$G_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset G_n(\mathbb{R}^{n+2}) \subset \dots$$

Example. $\mathbb{R}P^\infty = G_1 = G_1(\mathbb{R}^\infty)$ is the direct limit of $\mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \mathbb{R}P^3 \subset \dots$.

~~Def~~ We can build the bundle $\gamma^n$ over $G_n$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, hook] & E(\gamma^n) \subset G_n \times \mathbb{R}^\infty \arrow[d, "\pi"] \\
& G_n
\end{tikzcd}$$

$$E(\gamma^n) = \left\{ (X, \vec{x}) \;\middle|\; \begin{array}{l} X \text{ is an } n\text{-plane in } \mathbb{R}^\infty, \\ \vec{x} \in \mathbb{R}^\infty \text{ is a vector in } X \end{array} \right\}$$

This is called the "universal" n-plane bundle because:

Theorem (5.6) Any $\mathbb{R}^n$-bundle over a paracompact base space admits a bundle map $\xi \to \gamma^n$.

The key fact about the universal bundle will turn out to be:

**Theorem 5.7.** Any two bundle maps from $\xi$ to $\gamma^n$ are homotopic (through bundle maps).

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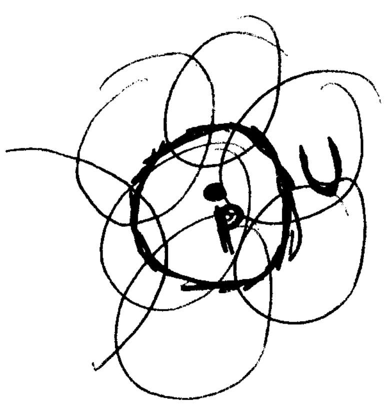
**Next time.** Proofs!

# Universal bundles, part II

We need to recall just a little bit about point-set topology here:

Definition. A space $X$ is paracompact if every open cover has a locally finite refinement.

Recall that locally finite $\Leftrightarrow$ each $p$ has an open neighborhood $U$ intersecting finitely many sets of the cover.




Your book uses "paracompact" to mean "paracompact Hausdorff".

Essentially everything is paracompact, Hausdorff.

Theorem (Stone) Every metric space is pcH.

Theorem (Morita) A regular space which is a countable union of compact sets is pcH.

Corollary. The direct limit of a sequence of compact Hausdorff spaces is paracompact.

Lemma. The infinite Grassmannian $G_n$ is pcH.

Proof. $G_n$ is the direct limit of

$$G_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset \dots \subset G_n(\mathbb{R}^{n+k}) \subset \dots$$

Theorem. Any $\mathbb{R}^n$-bundle $\xi$ over a paracompact Hausdorff base space admits a bundle map $\xi \to \gamma^n$.

Proof. Use paracompactness to deduce

Lemma. For any $\xi$ over a paracompact Hausdorff base, $\exists$ a locally finite covering by countably many open sets $U_i$ so that $\xi|_{U_i}$ is trivial.

Now mimic the proof from last lecture for compact base spaces. $\square$

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Now we have a more interesting

Theorem. Any two bundle maps from an $\mathbb{R}^n$-bundle to $\gamma^n$ are bundle-homotopic.

Proof. As before, a bundle map $f: \xi \to \gamma^n$ determines

$$\hat{f}: E(\xi) \to \mathbb{R}^\infty$$

where $\hat{f}|_{\text{any fiber}}$ is linear, injective.

Conversely, any such $\hat{f}$ determines $f$ by

$$f(e) = (\hat{f}(\text{fiber through } e), \hat{f}(e)).$$

Now we start to work. Let $f, g$ be any two bundle maps.

Case 1. $\hat{f}(e) \neq -\lambda^2 \hat{g}(e)$ for $e \neq 0 \in E(\xi)$

Then we can take the homotopy

$$\hat{h}_t(e) = (1-t)\hat{f}(e) + t\hat{g}(e).$$

With a little technical effort, one can show this is continuous. Further,

$$\hat{h}_t(e) \neq 0 \quad \text{for } e \neq 0$$

by assumption, and $\hat{h}_t(e)$ is linear on each fiber since $\hat{f}$ and $\hat{g}$ are.

So $\hat{h}_t$ induces a homotopy of bundle maps $h_t$, and it is not hard to prove continuity.

But why would Case 1 ever hold?
We intend to cheat.

Step 1. Observe that $d_1 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by $e_i \to e_{(2i-1)}$ is a linear map. This induces a bundle map $d_1 : \gamma^n \to \gamma^n$.

Let's use this to make $\hat{f}$ only nonzero on <u>odd</u> basis elements: $f \sim d_1 \circ f$ by Case 1.

Step 2. Similarly, let $d_2 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ take $e_i \to e_{2i}$, and observe $g \sim d_2 \circ g$ by Case 1.

Step 3. Now $d_1 \circ f \sim d_2 \circ g$ by Case 1.

(There's lots of room in $\mathbb{R}^\infty$!) $\square$

# Characteristic Classes.

**Lemma.** An $\mathbb{R}^n$-bundle $\xi$ over a pcH base $B$ determines a unique homotopy class of mappings $\bar{f}_\xi : B \to G_n$.

*Proof.* $\bar{f}_\xi$ is ~~thus~~ the base map of the bundle map $f_\xi : \xi \to \gamma^n$.

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So let

$$c \in H^i(G_n; R)$$

be any cohomology class. Any bundle $\xi$ determines a class

$$\bar{f}_\xi^{\,*} c \in H^i(B; R)$$

we call this class $c(\xi)$, the <u>characteristic class of $\xi$ determined by $c$.</u>

Now suppose we have a bundle map

$$
\begin{tikzcd}
\xi \arrow[rd, "f_\xi"] \arrow[d, "g"'] & \\
\eta \arrow[r, "f_\eta"'] & \gamma^n
\end{tikzcd}
$$

This diagram must homotopy-commute, so given any $c \in H^i(G_n; \mathbb{R})$, we must have

$$\bar{f}_\xi^* c = \bar{g}^* \circ \bar{f}_\eta^* c$$

or

$$c(\xi) = \bar{g}^* c(\eta)$$

So this construction obeys axiom 2 (naturality w.r.t. bundle maps).

On the other hand, given any construction

$$\xi \mapsto c(\xi) \in H^i(B(\xi); R)$$

which is natural w.r.t. bundle maps, we claim

$$c(\xi) = \bar{f}_\xi^* c(\gamma^n).$$

(This is obvious, since $f_\xi : \xi \to \gamma^n$ is a bundle map.)

This shows that the cohomology classes of $G_n$ determine

$\boxed{\text{all possible characteristic classes which are natural w.r.t. bundle maps}}$

Remark.

Actually, it can be proved that

$$\xi \cong \eta \iff f_\xi \sim f_\eta$$

so this is exactly enough information to determine the bundle.

Problems.

5B, 5C, 5D.

Next we try to determine the cohomology of $G_n$!