

Existence of SW classes.

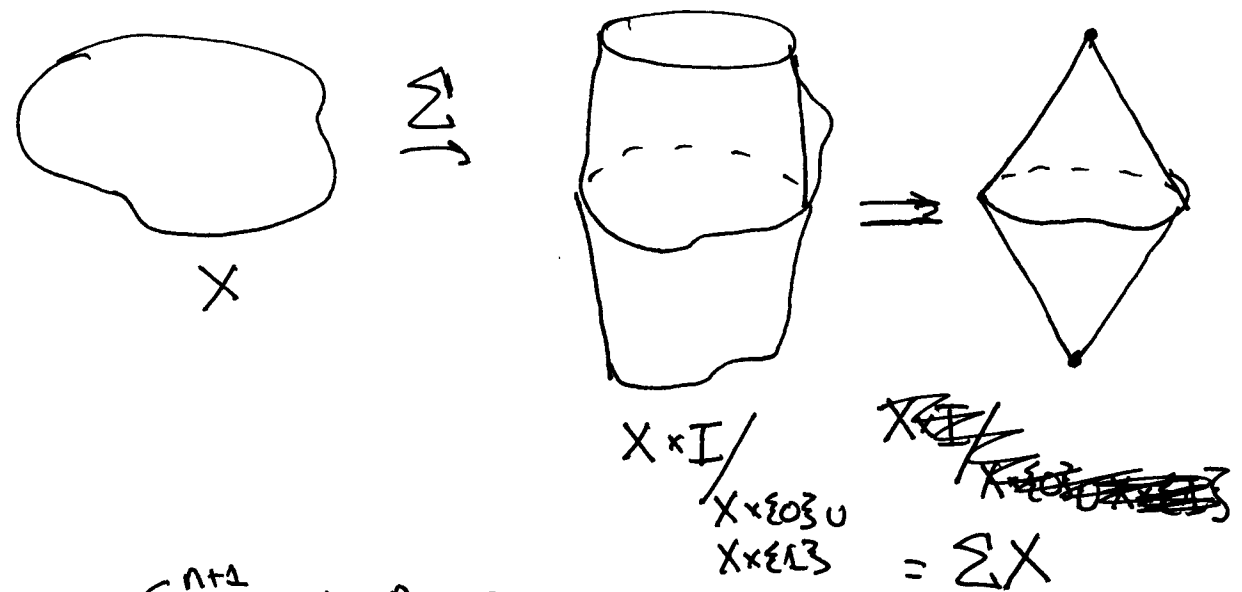
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Our goal is now to ~~den~~ prove that SW classes exist by giving an explicit construction in terms of known (if weird) cohomology operations.

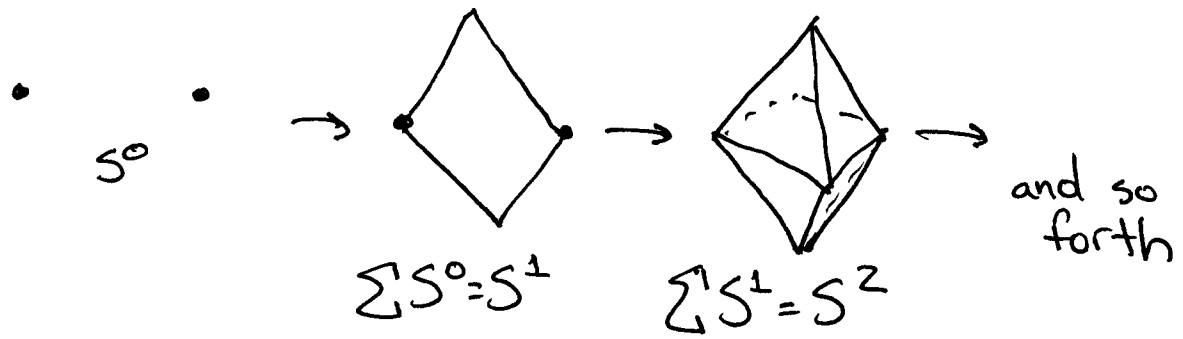
Some details we'll give today, others a few classes from now.

We start by recalling some stuff.

The operation "suspension" on a topological space X is given by



Example. $S^{n+1} = \Sigma S^n$ for all $n \geq 0$.



Theorem. There is an isomorphism

$$\sigma: H^n(X; \mathbb{Z}/2) \rightarrow H^{n+1}(\Sigma X; \mathbb{Z}/2)$$

(in singular homology, just suspend each map from a simplex into X , and all works out ok).

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Construction of Steenrod squares (after Hatcher)

We need to recall some wonderful stuff from algebraic topology.

Definition. The n -th homotopy group of a space X , denoted $\pi_n(X)$, is the ~~group~~ group of homotopy classes of maps $S^n \rightarrow X$.

Definition. The space $K(G, n)$ has $\pi_n = G$ and no other homotopy groups. It is called an Eilenberg-MacLane space.

The amazing fact about these is:

Theorem. There is a natural bijection

$$T: \langle X, K(G, n) \rangle \rightarrow H^n(X; G)$$

↑ homotopy classes of maps $X \rightarrow K(G, n)$.

where $T[f] = f^*(\alpha)$ for a certain class in $H_n(K(G, n); G)$.

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The details of this are very cool, but we don't want to focus on them now.

Instead, our goal is to use this theory to build a certain kind of operation on cohomology.

Definition. Steenrod squares $Sq^i: H^n(X; \mathbb{Z}/2) \rightarrow H^{n+i}(X; \mathbb{Z}/2)$ are cohomology operations which satisfy

1) $Sq^i(f^*(\alpha)) = f^*(Sq^i(\alpha))$ for $f: X \rightarrow Y$

2) $Sq^i(\alpha + \beta) = Sq^i(\alpha) + Sq^i(\beta)$

3) $Sq^i(\alpha \cup \beta) = \sum_j Sq^j(\alpha) \cup Sq^{i-j}(\beta)$.

4) $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where σ is the suspension isomorphism

5) $Sq^n(\alpha) = \alpha \cup \alpha$, $Sq^i(\alpha) = 0$ if $i > n$.

6) Sq^0 is the identity map

~~7)~~

We recall first that there is a natural cross product

$$\times: H^i(X) \times H^j(X) \rightarrow H^{i+j}(X \times X)$$

in cellular cohomology given by

$$(\alpha \times \beta)(\underset{c}{\cancel{e_i \times e_j}}) = \begin{cases} \alpha(e_\alpha^i) \beta(e_\beta^j), & \text{if } c = e_\alpha^i \times e_\beta^j \\ & \text{for } i \text{ and } j \\ & \text{cells in } X \\ 0 & \text{, if } c \text{ is not the} \\ & \text{product of } i \text{ and } j \\ & \text{cells in } X \end{cases}$$

With respect to this map, we can take the diagonal embedding $\Delta: X \rightarrow X \times X$ and define

$$\alpha \cup \beta = \Delta^*(\alpha \times \beta).$$

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Now we know that for $\alpha \in H^n(X; \mathbb{Z}/2)$,
 we ~~have~~ can write $\alpha \times \alpha \in H^{2n}(X \times X; \mathbb{Z}/2)$
 as a map $X \times X \rightarrow K(\mathbb{Z}/2, 2n)$, and hence

$$\begin{array}{ccc}
 \cancel{X \times X} & X \times X & \xrightarrow{[\alpha \times \alpha]} K(\mathbb{Z}/2, 2n) \\
 & \uparrow \Delta & \nearrow [\alpha \cup \alpha] \\
 & X &
 \end{array}$$

commutes, since $\Delta^*([\alpha \times \alpha]) = [\alpha \cup \alpha]$, and
 these classes are pullbacks of the
 same class in $K(\mathbb{Z}/2, 2n)$.

Now in $\mathbb{Z}/2$ coefficients, \cup and \times
 are commutative, so if

$$T: X \times X \rightarrow X \times X$$

takes (x_1, x_2) to (x_2, x_1) , then

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we have

$$T^*(\alpha \times \alpha) = \alpha \times \alpha.$$

But $\alpha \times \alpha$ represents a map $X \times X \rightarrow K(\mathbb{Z}/2, 2n)$
so $T^*(\alpha \times \alpha)$ represents a homotopic map
and \exists a homotopy f_t with

$$f_0 = \alpha \times \alpha, \quad f_1 = (\alpha \times \alpha) \circ T,$$

and $f_t: X \times X \times I \rightarrow K(\mathbb{Z}/2, 2n)$. Now clearly

$$T: X \times X \times I \rightarrow X \times X \times I \quad \text{as well,}$$

so

$$f_t \circ T: X \times X \times I \rightarrow K(\mathbb{Z}/2, 2n)$$

is a homotopy from $(\alpha \times \alpha) \circ T$ to $(\alpha \times \alpha)$.

Composing these yields a loop

$$S^1 \times X \times X \rightarrow K(\mathbb{Z}/2, 2n).$$

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Generally speaking, such a loop represents a class in $H^{2n}(S^2 \times X \times X)$, but the details of this process guarantee that we can arrange for the class to be 0, and hence that the map is null homotopic.

If we buy this, the map extends to

$$D^2 \times X \times X \rightarrow K(\mathbb{Z}/2, 2n)$$

and composing with T , a map

$$S^2 \times X \times X \rightarrow K(\mathbb{Z}/2, 2n)$$

but again this is null homotopic...

Eventually, we get a map

$$S^\infty \times X \times X \rightarrow K(\mathbb{Z}/2, 2n)$$

so that

$$(s, x_1, x_2) \text{ and } (-s, x_2, x_1) \rightarrow \text{same pt in } K(\mathbb{Z}/2, 2n).$$

We can now construct a map

$$S^\infty \times X \xrightarrow{I \times \Delta} S^\infty \times (X \times X) \rightarrow K(\mathbb{Z}/2, 2n)$$

so (s, x) and $(-s, x)$ go to the same pt.

There is then a quotient map

$$\mathbb{R}P^\infty \times X \longrightarrow K(\mathbb{Z}/2, 2n)$$

which represents a class in $H^{2n}(\mathbb{R}P^\infty \times X; \mathbb{Z}/2)$.

The Kunnetth formula implies that any such class can be written

$$[\beta] = \sum [\gamma] \times [\alpha_i], \quad [\gamma] \in H^{\mathbb{Z}/2, 2n-i}(\mathbb{R}P^\infty), \quad [\alpha_i] \in H^i(X).$$

but we know the cohomology of $\mathbb{R}P^\infty$ is generated by $\omega \in H^1(\mathbb{R}P^\infty)$, so

$$[\beta] = \sum \omega^{2n-i} \times \alpha_i$$

In this decomposition, there is a unique α_i in each dimension from 0 to $2n$. We take

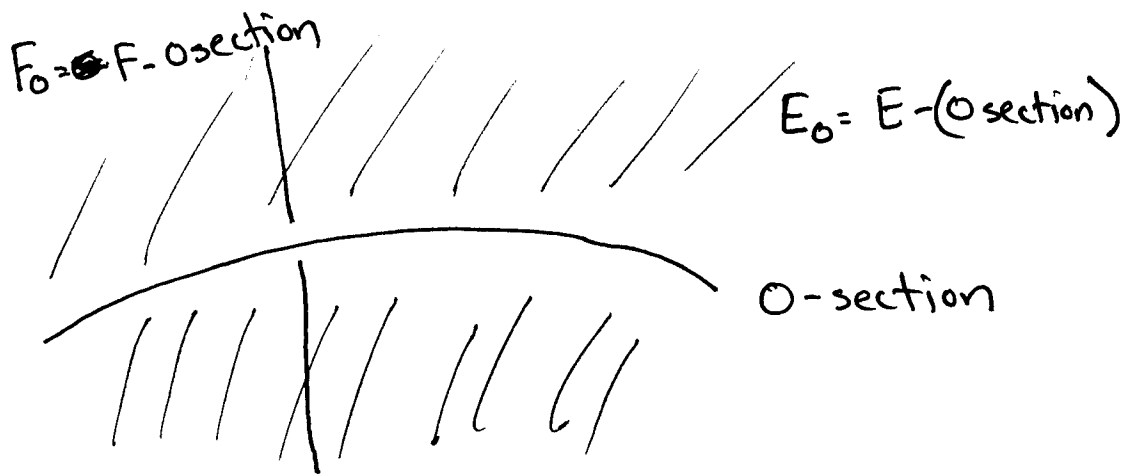
$$\alpha_{n+i} \in H^{n+i}(X) = Sq^i(\alpha). \quad \square$$

Note that α_{2n} which is paired with ω^0 in the Kunnetth formula is the original map $X \rightarrow K(\mathbb{Z}/2, 2n)$ (because a 0-class in $H^*(\mathbb{R}P^\infty)$ corresponds to the restriction of our map to the 0-cell in $\mathbb{R}P^\infty$ crossed with X), so this makes some sense.

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We now introduce the Thom class.

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Here's the basic idea. Given a vector bundle

$$\begin{array}{ccc} F^n & \rightarrow & E^{n+k} \\ & & \downarrow \\ & & B^k \end{array}$$

we can embed B in E as the 0-section.

Now if we compactify each fiber to a D^n , we have a disk bundle \tilde{E} over B^k .

In this bundle, in $\mathbb{Z}/2$ coefficients,

$$\text{pd: } H^k(B^k) \cong H^n(\tilde{E}, \partial\tilde{E})$$

by Poincaré duality.

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Since we're in $\mathbb{Z}/2$ coefficients, there is a top class for B^k called $[B^k]$.

Let

$$u \in H^n(\tilde{E}, \partial\tilde{E}) := \text{pd}([B^k]).$$

Definition. If we let $H^n(\tilde{E}, \partial\tilde{E}) \cong H^n(E, E_0)$ (since $\tilde{E}, \partial\tilde{E}$ is a deformation retract of E, E_0), then $u \in H^n(E, E_0)$ is the Thom class of the bundle E .

We will show:

Theorem. $H^i(E, E^0) = 0$ for $i < n$, and $u \in H^n(E, E^0)$ is the unique class so that $u|_F \in H^n(F, F^0)$ is the unique nonzero class in $H^n(F, F_0)$. Further,

$$\cup u: H^k(E) \rightarrow H^{k+n}(E, E_0)$$

is an isomorphism.

Now B is a deformation retract of E ,
so $H^k(B) \cong H^k(E)$ for all k , with
 $\pi^*: H^k(B) \rightarrow H^k(E)$ the isomorphism.

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Definition. The Thom isomorphism

$\varphi: H^k(B) \rightarrow H^{k+n}(E, E_0)$ is given by

$$x \mapsto \pi^*(x) \cup u.$$

We now ~~compute~~ define

Definition. The Stiefel-Whitney class

$w_i(\xi) \in H^i(B)$ is given by

$$w_i(\xi) = \varphi^{-1} Sq^i \varphi(1)$$

where $1 \in H^0(B)$ is the class which generates $H^0(B)$.

Let's parse that:

$$1 \in H^0(B).$$

$$\varphi(1) \in H^0(E, E_0) = u$$

$$Sq^i(\varphi(1)) \in H^i(E, E_0).$$

(yes, the squares work the same in relative cohomology)

$$\varphi^{-1}(Sq^i(\varphi(1))) \in H^i(B).$$

Now it is handy to have

$$Sq(\alpha) = \alpha + Sq^2(\alpha) + \dots + Sq^n(\alpha).$$

In this language,

$$Sq(\alpha \cup \beta) = (Sq \alpha) \cup (Sq \beta)$$

and

$$Sq(\alpha \times \beta) = (Sq \alpha) \times (Sq \beta).$$

With respect to this convention,

$$\omega(\xi) = \varphi^{-1} Sq \varphi(1) = \varphi^{-1} Sq(u).$$

So the Stiefel-Whitney classes are ~~projections~~ restrictions of the Steenrod squares of the Thom class of ξ .
classes that cup with u to get

Now we need only verify the axioms.

Axiom 1. $w_0 = 1$, $w_i = 0$ for $i > n$.

By Prop (5), $Sq^i(u) = 0$ for $i > n$. By (6),

$$Sq^0(u) = u = 1 \cup u.$$

Axiom 2. If $f: \xi \rightarrow \xi'$ is a bundle map, then $\bar{f}^*(w_i(\xi')) = w_i(\xi)$.

We need to have f induce a map

$$g: (E, E_0) \rightarrow (E', E'_0)$$

which is clearly true. The Thom class is natural in the sense that

$$g^*(u') = u$$

where u', u are the Thom classes in the bundles ξ and ξ' . Now we have

$$\begin{array}{ccc} H^k(B') & \xrightarrow{\varphi'} & H^{n+k}(E', E_0') \\ \downarrow \bar{f}^* & & \downarrow g^* \\ H^k(B) & \xrightarrow{\varphi} & H^{n+k}(E, E_0) \end{array}$$

where $\bar{f}: B \rightarrow B'$ is the base map of f . Now if we include $B \hookrightarrow E$ and $B' \hookrightarrow E'$ as the 0-sections, we get a diagram

$$\begin{array}{ccc} B' \hookrightarrow E' & & H^k(B') \xrightarrow{\pi^*} H^k(E') \\ \uparrow \bar{f} & \uparrow g & \Rightarrow \downarrow \bar{f}^* \quad \downarrow g^* \\ B \hookrightarrow E & & H^k(B) \xrightarrow{\pi^*} H^k(E) \end{array}$$

which clearly commutes (both as maps and in cohomology).

But then

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$$\begin{array}{ccc} H^k(E') & \xrightarrow{\cup u'} & H^{n+k}(E', E_0') \\ \downarrow g^* & & \downarrow g^* \\ H^k(E) & \xrightarrow{\cup u} & H^{n+k}(E, E_0) \end{array}$$

commutes because given $\alpha' \in H^k(E')$,

$$\begin{aligned} g^*(\alpha' \cup u') &= g^*(\alpha') \cup g^*(u') \\ &= g^*(\alpha') \cup u. \end{aligned}$$

So the original diagram commutes
and

$$\varphi_0 \bar{f}^* = g^* \circ \varphi'$$

Then we have

$$\begin{aligned} \varphi_0 \bar{f}^*(\omega_i(\xi')) &= \varphi_0 \bar{f}^*(\varphi'^{-1}(S q^i(u'))) \\ &= g^*(\varphi'(\varphi'^{-1}(S q^i(u')))) \\ &= g^*(S q^i(u')) \end{aligned}$$

$$\begin{aligned} &= Sq^i(g^*(u')) \quad (\text{naturality of } Sq^i) \\ &= Sq^i(u). \end{aligned}$$

Now φ is an isomorphism, so we can apply φ^{-1} to both sides to get

$$\begin{aligned} \bar{f}^*(\omega_i(\xi')) &= \varphi^{-1} \circ Sq^i(u) \\ &= \omega_i(\xi). \end{aligned}$$

Next class: Axioms 3 and 4!