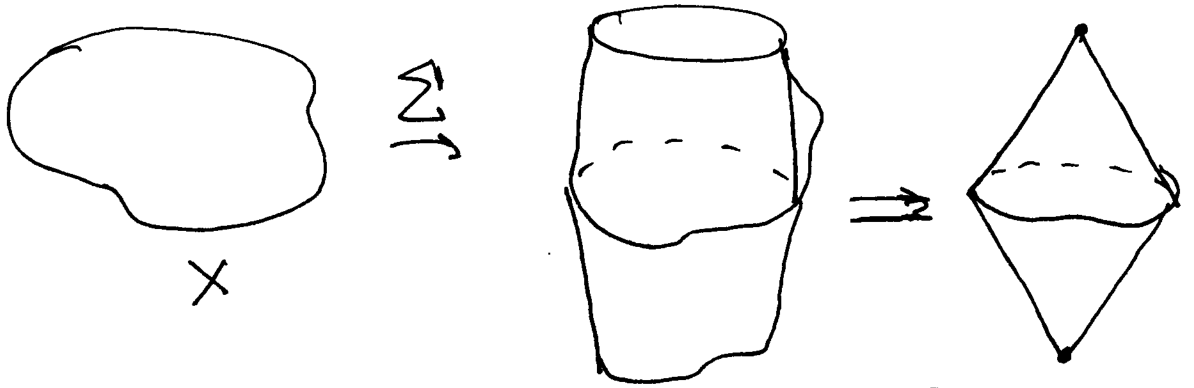
Existence of SW classes.

Our goal is now to prove that SW classes exist by giving an explicit construction in terms of known (if weird) cohomology operations. Some details we'll give today, others a few classes from now.

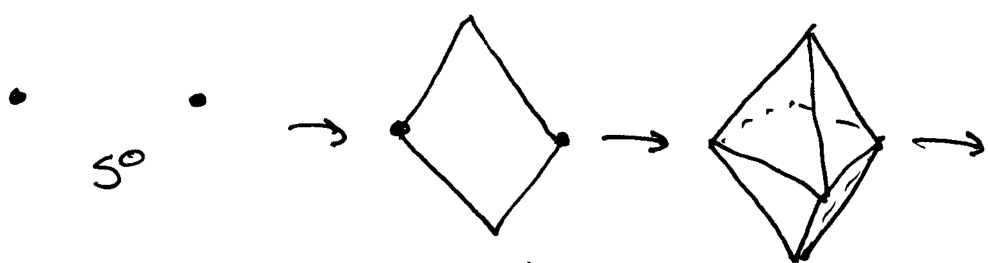
We start by recalling some stuff.

The operation "suspension" on a topological space X is given by



$$X \times I \big/ X\times\{0\} \cup X\times\{1\} = \Sigma X$$

Example. $S^{n+1} = \Sigma S^n$ for all $n \geq 0$.



Theorem. There is an isomorphism

$$\sigma : H^n(X; \mathbb{Z}/2) \to H^{n+1}(\Sigma X; \mathbb{Z}/2)$$

(in singular homology, just suspend each map from a simplex into X, and all works out ok).

Construction of Steenrod squares (after Hatcher)

We need to recall some wonderful stuff from algebraic topology.

Definition. The n-th homotopy group of a space X, denoted $\pi_n(X)$, is the group of homotopy classes of maps $S^n \to X$.

Definition. The space $K(G,n)$ has $\pi_n = G$ and no other homotopy groups. It is called an Eilenberg-MacLane space.

The amazing fact about these is:

Theorem. There is a natural bijection

$$T: \langle X, K(G,n) \rangle \to H^n(X;G)$$

homotopy classes of maps $X \to K(G,n)$.

where $T[f] = f^*(\alpha)$ for a certain class in $H_n(K(G,n);G)$.

The details of this are very cool, but we don't want to focus on them now. Instead, our goal is to use this theory to build a certain kind of operation on cohomology.

<u>Definition</u>. Steenrod squares $Sq^i : H^n(X;\mathbb{Z}/2) \to H^{n+i}(X;\mathbb{Z}/2)$ are cohomology operations which satisfy

1) $Sq^i(f^*(\alpha)) = f^*(Sq^i(\alpha))$ for $f: X \to Y$
2) $Sq^i(\alpha+\beta) = Sq^i(\alpha) + Sq^i(\beta)$
3) $Sq^i(\alpha \cup \beta) = \sum_j Sq^j(\alpha) \cup Sq^{i-j}(\beta)$.
4) $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where σ is the suspension isomorphism
5) $Sq^n(\alpha) = \alpha \cup \alpha$, $Sq^i(\alpha) = 0$ if $i > n$.
6) Sq^0 is the identity map

We recall first that there is a natural cross product

$$\times : H^i(X) \times H^j(X) \to H^{i+j}(X \times X)$$

in cellular cohomology given by

$$(\alpha \times \beta)(c) = \begin{cases} \alpha(e^i_\alpha)\,\beta(e^j_\beta), & \text{if } c = e^i_\alpha \times e^j_\beta \text{ for } i \text{ and } j \text{ cells in } X \\ 0, & \text{if } c \text{ is not the product of } i \text{ and } j \text{ cells in } X \end{cases}$$

With respect to this map, we can take the diagonal embedding $\Delta : X \to X \times X$ and <u>define</u>

$$\alpha \cup \beta = \Delta^*(\alpha \times \beta).$$

Now we know that ~~a~~ for $\alpha \in H^n(X;\mathbb{Z}/2)$, we ~~have~~ can write $\alpha \times \alpha \in H^{2n}(X \times X;\mathbb{Z}/2)$ as a map $X \times X \to K(\mathbb{Z}/2, 2n)$, and hence

$$\begin{tikzcd} X \times X \arrow[r, "{[\alpha \times \alpha]}"] & K(\mathbb{Z}/2, 2n) \\ X \arrow[u, "\Delta"] \arrow[ur, "{[\alpha \cup \alpha]}"'] & \end{tikzcd}$$

commutes, since $\Delta^*([\alpha \times \alpha]) = [\alpha \cup \alpha]$, and these classes are pull backs of the same class in $K(\mathbb{Z}/2, 2n)$.

Now in $\mathbb{Z}/2$ coefficients, $\cup$ and $\times$ are commutative, so if

$$T : X \times X \to X \times X$$

takes (x_1, x_2) to (x_2, x_1), then we have

$$T^*(\alpha \times \alpha) = \alpha \times \alpha.$$

But $\alpha \times \alpha$ represents a map $X \times X \to K(\mathbb{Z}/2, 2n)$ so $T^*(\alpha \times \alpha)$ represents a homotopic map and $\exists$ a homotopy f_t with

$$f_0 = \alpha \times \alpha, \qquad f_1 = (\alpha \times \alpha)(T),$$

and $f_t : X \times X \times I \to K(\mathbb{Z}/2, 2n)$. Now clearly

$$T : X \times X \times I \to X \times X \times I \quad \text{as well,}$$

so

$$f_t \circ T : X \times X \times I \longrightarrow K(\mathbb{Z}/2, 2n)$$

is a homotopy from $(\alpha \times \alpha)T$ to $(\alpha \times \alpha)$. Composing these yields a loop

$$S^1 \times X \times X \to K(\mathbb{Z}/2, 2n).$$

Generally speaking, such a loop represents a class in $H^{2n}(S^1 \times X \times X)$, but the details of this process guarantee that we can arrange for the class to be 0, and hence that the map is null homotopic.

If we buy this, the map extends to

$$D^2 \times X \times X \to K(\mathbb{Z}/2, 2n)$$

and composing with T, a map

$$S^2 \times X \times X \to K(\mathbb{Z}/2, 2n)$$

but again this is null homotopic...

Eventually, we get a map

$$S^\infty \times X \times X \to K(\mathbb{Z}/2, 2n)$$

so that

$$(s, x_1, x_2) \text{ and } (-s, x_2, x_1) \longrightarrow \text{same pt in } K(\mathbb{Z}/2, 2n).$$

We can now construct a map

$$S^\infty \times X \xrightarrow{I \times \Delta} S^\infty \times (X \times X) \longrightarrow K(\mathbb{Z}/2, 2n)$$

so (s,x) and $(-s,x)$ go to the same pt.

There is then a quotient map

$$\mathbb{R}P^\infty \times X \longrightarrow K(\mathbb{Z}/2, 2n)$$

which represents a class in $H^{2n}(\mathbb{R}P^\infty \times X; \mathbb{Z}/2)$.

The Kunneth formula implies that any such class can be written

$$[\beta] = \sum_i [\gamma] \times [\alpha_i], \quad [\gamma] \in H^{2n-i}(\mathbb{R}P^\infty), \; [\alpha_i] \in H^i(X).$$

but we know the cohomology of $\mathbb{R}P^\infty$ is generated by $\omega \in H^1(\mathbb{R}P^\infty)$, so

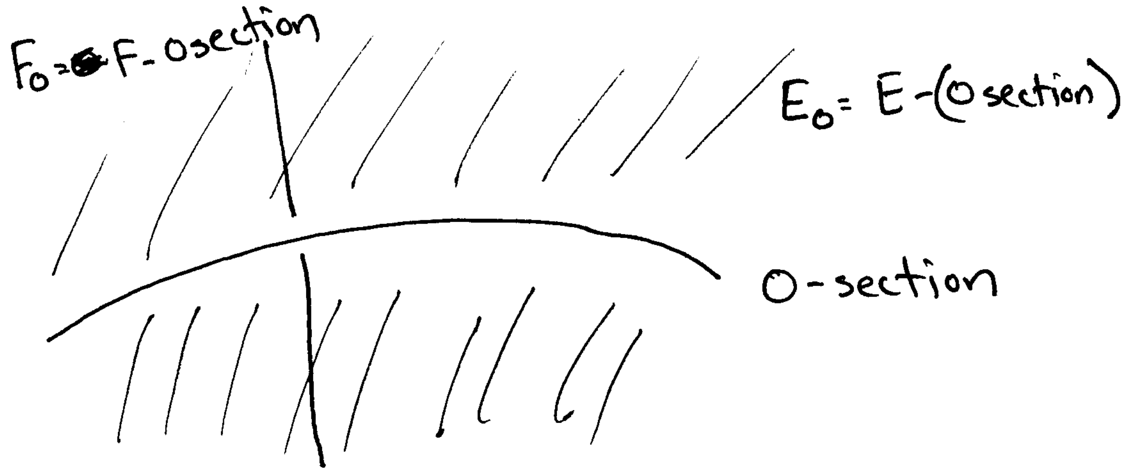
$$[\beta] = \sum_i \omega^{2n-i} \times \alpha_i$$

In this decomposition, there is a unique α_i in each dimension from 0 to $2n$. We take

$$\alpha_{n+i} \in H^{n+i}(X) = Sq^i(\alpha). \quad \square$$

Note that α_{2n}, which is paired with ω^0 in the Kunneth formula is the original map $X \to K(\mathbb{Z}/2, 2n)$ (because a 0-class in $H^*(\mathbb{RP}^\infty)$ corresponds to the restriction of our map to the 0-cell in $\mathbb{RP}^\infty$ crossed with X), so this makes some sense.

We now introduce the Thom class.



Here's the basic idea. Given a vector bundle

$$\begin{array}{ccc} F^n & \to & E^{n+k} \\ & & \downarrow \\ & & B^k \end{array}$$

we can embed B in E as the 0-section. Now if we compactify each fiber to a D^n, we have a disk bundle $\tilde{E}$ over B^k. In this bundle, in $\mathbb{Z}/2$ coefficients,

$$pd: H^k(B^k) \cong H^n(\tilde{E}, \partial\tilde{E})$$

by Poincare duality.

Since we're in $\mathbb{Z}/2$ coefficients, there is a top class for B^k called $[B^k]$. Let

$$u \in H^n(\tilde{E}, \partial\tilde{E}) := pd([B^k]).$$

<u>Definition</u>. If we let $H^n(\tilde{E}, \partial\tilde{E}) \cong H^n(E, E_0)$ (since $\tilde{E}, \partial\tilde{E}$ is a deformation retract of E, E_0), then $u \in H^n(E, E_0)$ is the <u>Thom class</u> of the bundle E.

We will show:

Theorem. $H^i(E, E^0) = 0$ for $i < n$, and $u \in H^n(E, E^0)$ is the unique class so that $u|_F \in H^n(F, F^0)$ is the unique nonzero class in $H^n(F, F_0)$. Further,

$$\cup u : H^k(E) \to H^{k+n}(E, E_0)$$

is an isomorphism.

Now B is a deformation retract of E, so $H^k(B) \cong H^k(E)$ for all k, with $\pi^*: H^k(B) \to H^k(E)$ the isomorphism.

<u>Definition.</u> The <u>Thom isomorphism</u> $\varphi: H^k(B) \to H^{k+n}(E, E_0)$ is given by $x \mapsto \pi^*(x) \cup u$.

We now ~~compute~~ define

<u>Definition.</u> The Stiefel-Whitney class $w_i(\xi) \in H^i(B)$ is given by

$$w_i(\xi) = \varphi^{-1} Sq^i \varphi(1)$$

where $1 \in H^0(B)$ is the class which generates $H^0(B)$.

Let's parse that:

$$1 \in H^0(B).$$

$$\varphi(1) \in H^0(E, E_0) = u$$

$$Sq^i(\varphi(1)) \in H^i(E, E_0).$$

(yes, the squares work the same in relative cohomology)

$$\varphi^{-1}(Sq^i(\varphi(1))) \in H^i(B).$$

Now it is handy to have

$$Sq(\alpha) = \alpha + Sq^1(\alpha) + \dots + Sq^n(\alpha).$$

In this language,

$$Sq(\alpha \cup \beta) = (Sq\,\alpha) \cup (Sq\,\beta)$$

and

$$Sq(\alpha \times \beta) = (Sq\,\alpha) \times (Sq\,\beta).$$

With respect to this convention,

$$w(\xi) = \varphi^{-1} Sq\,\varphi(1) = \varphi^{-1} Sq(u).$$

So the Stiefel-Whitney classes are ~~projections~~ restrictions of the Steenrod
classes that cup with u to get
squares of the Thom class of ξ.

Now we need only verify the axioms.

Axiom 1. $w_0 = 1$, $w_i = 0$ for $i > n$.

By Prop (5), $Sq^i(u) = 0$ for $i > n$. By (6), $Sq^0(u) = u = 1 \cup u$.

Axiom 2. If $f: \xi \to \xi'$ is a bundle map, then $\bar{f}^*(w_i(\xi')) = w_i(\xi)$.

We need to have f induce a map

$$g: (E, E_0) \to (E', E_0')$$

which is clearly true. The Thom class is natural in the sense that

$$g^*(u') = u$$

where u', u are the Thom classes in the bundles ξ and ξ'. Now we have

$$\begin{array}{ccc} H^k(B') & \xrightarrow{\varphi'} & H^{n+k}(E', E_0') \\ \downarrow \bar{f}^* & & \downarrow g^* \\ H^k(B) & \xrightarrow{\varphi} & H^{n+k}(E, E_0) \end{array}$$

where $\bar{f}: B \to B'$ is the base map of f. Now if we include $B \hookrightarrow E$ and $B' \hookrightarrow E'$ as the 0-sections, we get a diagram

$$\begin{array}{ccc} B' & \hookrightarrow & E' \\ \uparrow \bar{f} & & \uparrow g \\ B & \hookrightarrow & E \end{array} \quad \Rightarrow \quad \begin{array}{ccc} H^k(B') & \xrightarrow{\pi^*} & H^k(E') \\ \downarrow \bar{f}^* & & \downarrow g^* \\ H^k(B) & \xrightarrow{\pi^*} & H^k(E) \end{array}$$

which clearly commutes (both as maps and in cohomology).

But then

$$\begin{tikzcd} H^k(E') \arrow[r, "\cup u'"] \arrow[d, "g^*"] & H^{n+k}(E', E_0') \arrow[d, "g^*"] \\ H^k(E) \arrow[r, "\cup u"] & H^{n+k}(E, E_0) \end{tikzcd}$$

commutes because given $\alpha' \in H^k(E')$,

$$g^*(\alpha' \cup u') = g^*(\alpha') \cup g^*(u')$$
$$= g^*(\alpha') \cup u.$$

So the original diagram commutes and

$$\varphi \circ \bar{f}^* = g^* \circ \varphi'$$

Then we have

$$\varphi \circ \bar{f}^*(w_i(\xi')) = \varphi \circ \bar{f}^*(\varphi'^{-1}(Sq^i(u')))$$
$$= g^*(\varphi'(\varphi'^{-1}(Sq^i(u'))))$$
$$= g^*(Sq^i(u'))$$

$$= Sq^i(g^*(u')) \quad \text{(naturality of } Sq^i\text{)}$$

$$= Sq^i(u).$$

Now φ is an isomorphism, so we can apply φ^{-1} to both sides to get

$$\bar{f}^*(w_i(\xi')) = \varphi_{\alpha}^{-1}(Sq^i(u))$$

$$= w_i(\xi).$$

Next class: Axioms 3 and 4!

Existence of SW classes (II)

Axiom 3. The Whitney Product theorem.

If ξ, η are vector bundles over B,

$$w_k(\xi \oplus \eta) = \sum_i w_i(\xi) \cup w_{k-i}(\eta).$$

We start by considering a product bundle $\xi'' = \xi \times \xi'$ given by

$$\begin{array}{c} E \times E' = E'' \\ \big\downarrow \pi \times \pi' \\ B \times B' \end{array}$$

There are Thom classes $u \in H^m(E, E_0)$, $u' \in H^n(E', E_0')$

It turns out that we can define a relative cross product on these groups so that

$$u \times u' \in H^{m+n}(E \times E', (E \times E_0') \cup (E_0 \times E'))$$

$$\in H^{m+n}(E'', E_0'').$$

We claim that

$$U \times U' = u'' \in H^{m+n}(E'', E''_0),$$

the Thom class of the product bundle. To prove it, we only have to show

$$U \times U' \,|\, (F'', F''_0)$$

is the nonzero class in $H^{m+n}(F'', F''_0)$. But $F'' = F \times F'$, so this is clear. We recall that

$$(a \times b) \cup (c \times d) = (a \cup c) \times (b \cup d)$$

so

$$\begin{aligned} \varphi''(a \times b) &= (U \times U') \cup (a \times b) \\ &= (u \cup a) \times (u' \cup b) \\ &= \varphi(a) \times \varphi'(b). \end{aligned}$$

Now the total SW class of ξ'' is given by

$$\varphi''(w(\xi'')) = Sq(u'') = Sq(u \times u') = Sq(u) \times Sq(u')$$

(the last step requires that you believe that the Sq operation construction splits in the expected way under $\times$. It does.)

Now

$$Sq(u)\times Sq(u') = \varphi(\omega(\xi))\times\varphi'(\omega(\xi'))$$
$$= \varphi''(\omega(\xi)\times\omega(\xi'))$$

We know φ'' is an isomorphism, so this implies

$$\omega(\xi'') = \omega(\xi)\times\omega(\xi').$$

Now $\xi\oplus\xi'$ is the sub-bundle of $\xi\times\xi'$ obtained by restricting to the diagonal of $B\times B$. The inclusion is a bundle map, so

$$\Delta^*(\omega(\xi'')) = \omega(\xi\oplus\xi').$$

and

$$\omega(\xi\oplus\xi) = \Delta^*(\omega(\xi)\times\omega(\xi'))$$

Now what does

$$\omega(\xi) \times \omega(\xi') \quad \underline{\text{mean}}?$$

Well, it really means

$$\sum_{i,j} \omega_i(\xi) \times \omega_j(\xi'), \quad \text{a sum of classes in various dimensions}$$

In a particular dimension k, this is

$$\sum_i \omega_i(\xi) \times \omega_{k-i}(\xi')$$

so

$$\Delta^*\left(\sum_i \omega_i(\xi) \times \omega_{k-i}(\xi')\right) = \sum_i \Delta^*(\omega_i(\xi) \times \omega_{k-i}(\xi'))$$

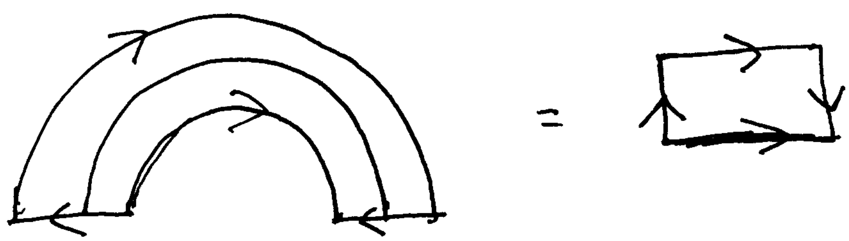
$$= \sum_i \omega_i(\xi) \cup \omega_{k-i}(\xi')$$

by the definition of cup. This is what we wanted to show! $\square$

We now prove axiom 4.

Axiom 4. $w_1(\gamma_1^1) \neq 0$.

Recall γ_1^1 is the twisted line bundle over $\mathbb{RP}^1 = S^1$. If we take vectors of length ≤ 1 in $E(\gamma_1^1)$, we get



a mobius band M, bounded by a circle $\dot{M}$.
Now (E, E_0) deformation retracts to $(M, \dot{M})$, so

$$H^*(E, E_0) \cong H^*(M, \dot{M}).$$

Further,

$$\mathbb{RP}^2 = \text{mobius band} \cup_{\dot{M}} \text{disk},$$

so by excision,

$$H^*(M, \dot{M}) = H^*(\mathbb{RP}^2 - \text{int}\, D^2, D^2 - \text{int}\, D^2)$$
$$= H^*(\mathbb{RP}^2, D^2).$$

reduced!

Now the long exact sequence of the pair $(\mathbb{RP}^2, D^2)$ yields

$$\left(0 = \tilde{H}^i(D^2)\right) \leftarrow \tilde{H}^i(\mathbb{RP}^2) \xleftarrow{\cong} \tilde{H}^i(\mathbb{RP}^2, D^2) \leftarrow \left(\tilde{H}^{i-1}(D^2) = 0\right)$$

so we have isomorphisms

$$H^1(E, E_0) \cong H^1(M, \dot{M}) \cong H^1(\mathbb{RP}^2, D^2) \cong H^1(\mathbb{RP}^2)$$

but $H^1(\mathbb{RP}^2) \neq 0$, and is generated by a.

~~Now the same holds in H^2, so~~ So $H^1(E, E_0)$ is nonzero and generated by u, which is the Thom class of γ_1^1.

Now

$$a \cup a \neq 0 \text{ in } H^2(\mathbb{RP}^2)$$

so

$$u \cup u \neq 0 \text{ in } H^2(E, E_0)$$

(since the isomorphism holds in $i = 2$).

Now

$$u \cup u = Sq^1(u),$$

so $Sq^1(u) \neq 0$. This means

$$w_1(\gamma_1^1) = \varphi^{-1} Sq^1(u) \neq 0,$$

since φ^{-1} is an isomorphism. $\square$

We have now proved that SW classes exist in $\mathbb{Z}/2$ coefficients! Our next version of the theory will work in $\mathbb{Z}$ coefficients, (at the price of making everything orientable) and give more information.