Oriented Bundles and the Euler class

Recall:

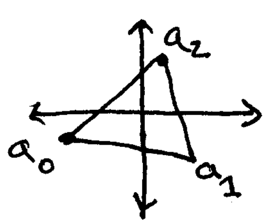
Definition. An <u>orientation</u> on a real vector space of dimension $n>0$ is an equivalence class of bases where

$$v_1,\dots,v_n \cong v_1',\dots,v_n' \iff \text{the matrix } \vec{v}_i' = A\vec{v}_i \text{ has positive determinant.}$$

Our goal is to switch from $\mathbb{Z}/2$ to $\mathbb{Z}$ coefficients. To do this, we need to add the hypothesis that everything is orientable.

We first recall how orientations are defined for vector bundles from the definition for vector spaces.

1) A simplex Δ^n linearly embedded in V defines an orientation for V.



$$\Delta^2 \longrightarrow (a_0a_1, a_0a_2)$$

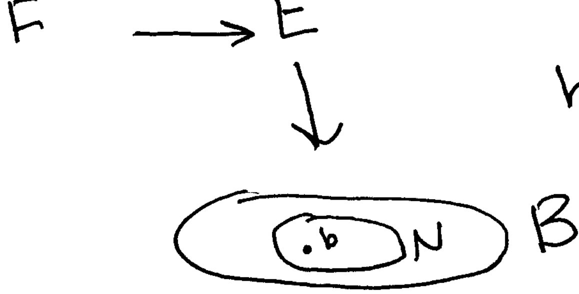
2) The map $\sigma : \Delta^n \to V$ represents in $H_n(V, V_0; \mathbb{Z})$ if $\vec{0}$ inside im σ. In fact, σ is one of the two generators of this group.

3) We call μ_V the preferred generator for $H_n(V, V_0; \mathbb{Z})$ according to the orientation.

4) There is a corresponding generator u_V for $H^n(V, V_0; \mathbb{Z})$ defined by $\langle u_V, \mu_V \rangle = +1$.

We can now define:

Definition. An <u>orientation</u> for a vector bundle ξ is a function which gives an orientation to each fiber of ξ so that in a localtrivialization (N, h)



$h : N \times \mathbb{R}^n \to \pi^{-1}(N)$

We have an orientation on the $\mathbb{R}^n$ so that

$$x \mapsto h(b, x) \text{ is o.p. for each } b \in N.$$

We note that it's the same to require that the orientation on each fiber is determined by n local sections $s_1, \dots, s_n$ of ξ over N.

We can now state an oriented version of the Thom theorem:

Theorem. Let ξ be an oriented n-plane bundle $F \to E \to B$. Then

$$H^i(E, E_0; \mathbb{Z}) = \begin{cases} 0, & \text{for } i < n \\ \text{contains one and only one class } u \text{ as below}, & \text{for } i = n \end{cases}$$

where

i) $u|_{(F, F_0)}$ is the preferred generator for $H^n(F, F_0; \mathbb{Z})$

ii) $y \mapsto y \cup u$ is an isomorphism $H^k(E; \mathbb{Z}) \to H^{n+k}(E, E_0; \mathbb{Z})$

As before,

u is the Thom class

$\varphi: H^k(B;\mathbb{Z}) \to H^{k+n}(E,E_0;\mathbb{Z})$ is the Thom isomorphism

We can now define a new and cool characteristic class.

$$(E,\emptyset) \hookrightarrow (E,E_0)$$

yields a map

$$H^*(E,E_0) \to H^*(E)$$

So the image of the Thom class is a new class $u \in H^n(E;\mathbb{Z}) \cong H^n(B;\mathbb{Z})$.

Definition. We call this the Euler class $e(\xi)$ of ξ.

We have the following properties:

Naturality. If $f: B \to B'$ is the base map of an orientation preserving bundle map $\xi \to \xi'$ then $e(\xi) = f^* e(\xi')$.

Remark. If ξ is trivial, this means $e(\xi) = 0$.

Orientation. If $f: \xi \to \xi$ is orientation-reversing, then $f^* e(\xi) = -e(\xi)$.

Odd bundles. If n is odd, for an n-plane bundle

$$e(\xi) + e(\xi) = 0.$$

Proof. There is still a Thom image

$$\begin{aligned} \varphi(e(\xi)) &= \pi^*(e(\xi)) \cup u \\ &= u|_E \cup u \\ &= u \cup u. \end{aligned}$$

Why? Well,

$$\begin{tikzcd} H^n(E) \arrow[dr, "\cup u"'] & H^n(E, E_0) \arrow[l] \arrow[d, "\cup u"] \\ & H^{n+n}(E, E_0) \end{tikzcd}$$

ought to commute (by ~~ξ~~ naturality of cup?). We could go back to definitions, I guess. It's ok. from the p.o.v. of forms, and at a chain level.

Thus

$$e(\xi) = \varphi^{-1}(u \cup u)$$

But $u \cup u = (-1)^{n^2} u \cup u = -u \cup u$ if n is odd. □.

Proposition. The homomorphism $H^n(B;\mathbb{Z}) \to H^n(B;\mathbb{Z}/2)$ induced by $\mathbb{Z} \to \mathbb{Z}/2$ carries $e(\xi) \mapsto w_n(\xi)$.

Proof. Apply this to

$$\begin{array}{ccc} e(\xi) & = & \varphi^{-1}(u \cup u) \\ \downarrow & & \downarrow \\ w_n(\xi) & = & \varphi^{-1}(Sq^n(u)) \end{array}$$

Keeping in mind that the integer Thom class maps to the $\mathbb{Z}/2$ Thom class. □

Proposition. For a Whitney sum,

$$e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$$

and for a product

$$e(\xi \times \xi') = e(\xi) \times e(\xi').$$

Proof. We claim that if ξ an m-bundle, ξ' an n-bundle

$$u(\xi \times \xi') = (-1)^{mn} u(\xi) \times u(\xi').$$

Now applying

$$H^{m+n}(E \times E', (E \times E')_0) \to H^{m+n}(E \times E') \cong H^{m+n}(B \times B')$$

to both ~~~~ sides, and observing that this homomorphism splits into the restrictions

$$H^m(E, E_0) \to H^m(E) \quad \times \quad H^n(E', E'_0) \to H^n(E')$$

we get

$$e(\xi \times \xi') = (-1)^{mn} e(\xi) \times e(\xi').$$

Now if m or n is odd, then the rhs is equal to $-$ the rhs, so we can ignore the sign here.

This proves part ~~2~~. To see part 2, pull back ~~$H^{m+n}(B\times B)$~~ $H^{m+n}(B\times B)$ to $H^{m+n}(B)$ by Δ^* as usual. $\square$.

Note: This is less powerful than the formula $w(\xi\oplus\xi') = w(\xi)\cup w(\xi')$ since we can't generally solve for $e(\xi')$ as a function of $e(\xi)$ and $e(\xi\oplus\xi')$, since $e(\xi)$ may not be a unit in $H^*(B)$.

Example. If $e(\xi)\neq 0$, then ξ can't split as the sum of two oriented odd dimensional bundles.

Proposition. If ξ is an oriented Euclidean vector bundle with a nonzero cross section, then $e(\xi)=0$.

Proof. Let s be the section, $\xi = s\oplus s^{\perp}$; where s is trivial line bundle from section.

Then

$$e(\xi) = e(s) \cup e(s^{+})$$
$$= 0 \cup e(s^{+}).$$

Problem (9C.)