We now consider $G_1(\mathbb{C}^{k+1}) = \mathbb{C}P^k$.

Let $\gamma^1(\mathbb{C}^{k+1})$ be the canonical line bundle over $\mathbb{C}P^k$ and use the Gysin sequence on this (real 2-bundle) bundle

$$\to H^{i+1}(E_0) \to H^{i+1-2+1}(\mathbb{C}P^k) \xrightarrow{\cup e} H^{i+2}(\mathbb{C}P^k) \xrightarrow{\pi_0^*} H^{i+2}(E_0) \to$$

Now $e(\gamma^1_{\mathbb{R}})$ = the Chern class $c_1(\gamma^1)$. Further,

E_0 = set of pairs (line, nonzero vector on line)

(deformation retracts to) $S^{2k+1} \subset \mathbb{C}^{k+1} = \mathbb{R}^{2k+2}$

So our Gysin sequence is just a collection of isomorphisms...

$$H^i(\mathbb{C}P^k) \underset{\cup e}{\cong} H^{i+2}(\mathbb{C}P^k)$$

Now we know that

$$c_1(\gamma^1_{\cancel{\mathbb{R}}}) = e(\gamma^1_R)$$

(since γ^1 is a complex 1-plane bundle) so these isomorphisms are really

$$H^i(\mathbb{C}P^k) \underset{\cong}{\overset{\cup c_1}{\longleftrightarrow}} H^{i+2}(\mathbb{C}P^k)$$

and

$$H^0(\mathbb{C}P^k) \cong H^2(\mathbb{C}P^k) \cong \cdots \cong H^{2k}(\mathbb{C}P^k),$$

with ~~each~~ group $2i$ a $\mathbb{Z}$ generated by $c_1(\gamma^1)^i$. Now ~~ha~~we also have

$$H^1(\mathbb{C}P^k) \cong \cdots = H^{2k-1}(\mathbb{C}P^k)$$

by the same argument. We now compute $H^1(\mathbb{C}P^k)$.

at the bottom end of the sequence, we have

$$(\to H^0(E_0)) \to H^{-1}(\mathbb{C}P^k) \to H^1(\mathbb{C}P^k) \xrightarrow{\pi_0^*} H^1(E_0) \to$$

or

$$0 \to H^1(\mathbb{C}P^k) \xrightarrow{\pi_0^*} H^1(E_0) \to \$$$

But again, $E_0 = S^{2k+1}$, so $H^1(E_0) = 0$, and that means $H^1(\mathbb{C}P^k) = 0$.

We now know the entire cohomology ring of $\mathbb{C}P^k$. As before, we now let $k \to \infty$ to form $\mathbb{C}P^\infty = G_1(\mathbb{C}^\infty)$.

We see that

$H^*(G_1(\mathbb{C}^\infty))$ is the polynomial ring generated by $c_1(\gamma^1)$.

we claim

Theorem. $H^*(G_n(\mathbb{C}^\infty), \mathbb{Z})$ is the polynomial ring generated by Chern classes $c_1(\gamma^n), \ldots, c_n(\gamma^n)$. There are no polynomial relations between these classes.

Proof. Consider the canonical n-plane bundle γ^n over G_n in $\mathbb{C}^\infty$. We have a Gysin sequence

$$\to H^i(G_n) \xrightarrow{\cup c_n} H^{i+2n}(G_n) \xrightarrow{\pi_0^*} H^{i+2n}(E_0) \to H^{i+1}(G_n) \to$$

Now we want to show that $H^*(E_0) = H^*(G_{n-1})$. We start by building a map $f: E_0 \to G_{n-1}$.

A point in $E_0(\gamma^n)$ is in the form (X,v) where $v \in X$. Take

$$f(X,v) = X \cap v^{\perp},$$

the complement of v in X. This is an $n-1$ plane in $\mathbb{C}^{\infty}$.

We claim f induces cohomology isomorphisms. Consider $\gamma^n(\mathbb{C}^N) \subset \gamma^n$ for some large finite N, and let f_N be the restriction of f. Now given $Y \in G_{n-1}(\mathbb{C}^n)$, we see

$$f_N^{-1}(Y) \subset E_0(\gamma^n(\mathbb{C}^N))$$

is the set of pairs X, v where $v \perp Y$ and $X = Y + \mathbb{C}v$.

We use this as the projection in a new vector bundle

$\omega^{\underset{N-(n-1)}{\overset{\|}{(N-n)+1}}}$ over $G_{n-1}(\mathbb{C}^N)$ where

$$\begin{array}{c} E_0(\omega^{N-n+1}) \\ \big\downarrow f_N \\ G_{n-1}(\mathbb{C}^N) \end{array}$$

and $F_\Psi(\omega^{(N-n)+1})$ is the orthogonal complement of Ψ in $\mathbb{C}^N$. Now the Gysin sequence of <u>this</u> bundle is

$$\to H^{i-2(N-(n-1))}(G_{n-1}(\mathbb{C}^N)) \to H^i(G_{n-1}(\mathbb{C}^N)) \xrightarrow{\pi_0^*} H^i(E_0) \to H^{i-2(N-(n-1))+1}(G_{n-1})$$

(where we remember that ω is an $N-(n-1)$ plane bundle over G_{n-1})

Thus for

$$i < 2(N-(n-1))$$
$$< 2(N-n)+2$$

or $i \leq 2(N-n)$, this sequence shows that $\pi_0^* = f_N^*$ induces cohomology isomorphisms. But f is the direct limit of the f_N, and G_{n-1}, γ^n are the direct limits of $G_{n-1}(\mathbb{C}^N)$, $\gamma^n(\mathbb{C}^N)$, so this shows f induces cohomology isomorphisms in all dimensions

We now return to our original Gysin sequence

$$\begin{tikzcd}
\arrow[r] & H^i(G_n) \arrow[r, "\cup c_n"] & H^{i+2n}(G_n) \arrow[r, "\pi_0^*"] \arrow[dr, "\lambda"'] & H^{i+2n}(E_0) \arrow[r] & H^{i+1}(G_n) \arrow[r] & {} \\
& & & H^{i+2n}(G_{n-1}) \arrow[u, "f^*"'] \arrow[ur] & &
\end{tikzcd}$$

and modify it to a new sequence by inserting $H^{i+2n}(G_{n-1})$.

Note that the new map λ is, by construction, $(f^*)^{-1}\pi_0^*$.

We claim $\lambda(c_i(\gamma^n)) = c_i(\gamma^{n-1})$. For $i=n$, we recall that $c_i(\gamma^{n-1})=0$, since γ^{n-1} is an $n-1$ plane bundle and that our sequence reads

$$\to H^0(G_n) \xrightarrow{\cup c_n} H^{2n}(G_n) \xrightarrow{\lambda} H^{2n}(G_{n-1}) \to$$

thus by exactness, $\lambda(c_n(\gamma^n)) = 0 = c_n(\gamma^{n-1})$.

So suppose $i<n$. By definition,

$$c_i(\gamma_0^n) = \pi_0^* c_i(\gamma^n)$$

But γ_0^n, the "orthogonal bundle" over E_0, has

$$\begin{array}{ccc} E(\gamma_0^n) & & \gamma^{n-1} \\ \downarrow & & \downarrow \\ E_0(\gamma^n) & \xrightarrow{f} & G_{n-1} \end{array}$$

We claim f is covered by a bundle map $\hat{f}$. This is just a matter of unwinding definitions:

$$E(\gamma_0) = \{((X,\vec{v}), \vec{\omega}) \mid \vec{v} \in X, \vec{\omega} \in X, \vec{\omega} \perp \vec{v}\}$$

$$f(X,\vec{v}) = Y, \quad \text{where } Y = \vec{v}^{\perp} \cap X.$$

$$E(\gamma^{n-1}) = \{(Y,\vec{u}) \mid \vec{u} \in Y\}$$

so

$$\hat{f}((X,\vec{v}),\vec{\omega}) = (\vec{v}^{\perp} \cap X, \vec{\omega}).$$

Now by naturality of Chern classes, we have

$$\cancel{f^*(c_i(\gamma^{n-1})) = c_i(E_0(\gamma}$$

$$f^* c_i(\gamma^{n-1}) = c_i(\gamma_0^{n-1})$$

So now we have

$$f^* c_i(\gamma^{n-1}) = c_i(\gamma_0^{n-1}) = \pi_0^* c_i(\gamma^n)$$

~~but f^* E~~
applying $(f^*)^{-1}$ to both sides, since f^* is an isomorphism,

$$c_i(\gamma^{n-1}) = (f^*)^{-1} \pi_0^* c_i(\gamma^n)$$
$$= \lambda c_i(\gamma^n)$$

as claimed.

Now we have the new sequence, and we know λ carries Chern classes to Chern class. We now proceed by induction (on n):

$n=1$. $H^*(G_1) = H^*(\mathbb{C}P^\infty)$ is generated by the Chern class c_1.

Now suppose this statement holds (31)
for $n-1$, so $H^*(G_{n-1})$ is generated by
Chern classes, Then in our sequence
and each elt is expressed uniquely as a poly.

$$\xrightarrow{\lambda} H^{i+2n-1}(G_{n-1}) \longrightarrow H^i(G_n) \xrightarrow{\cup c_n} H^{i+2n}(G_n) \xrightarrow{\lambda} H^{i+2n}(G_{n-1}) \longrightarrow$$

the λ terms are surjective so we have

$$0 \xrightarrow{0} H^i(G_n) \xrightarrow{\cup c_n} H^{i+2n}(G_n) \xrightarrow{\lambda} H^{i+2n}(G_{n-1}) \longrightarrow 0$$

We want to show by induction on i that each $x \in H^{i+2n}(G_n)$ can be written uniquely as ~~an~~ polynomial in the c_i.

This is a standard argument from injectivity of ~~the~~ the $\cup c_n$ map and induction (we worked out details on the fly).

□

Induced and Universal Bundles over $\mathbb{C}$

We previously had the statement "every real n-bundle has a ~~map~~ bundle map into γ^n" and "two real bundles ~~which~~ have homotopic bundle maps into γ^n <=> they are isomorphic as bundles". Even this can be sharpened to

"two real bundles are isomorphic <=> their base maps into $G_n^{\#}$ are homotopic"!

We now define:

Definition. Given a map $f: B \to G_n$, the induced bundle $f^*(\gamma^n)$ is defined by

$$\begin{tikzcd} E(f^*(\gamma^n)) \subset B \times \mathbb{R}^\infty \arrow[r] \arrow[d] & E(\gamma^n) \arrow[d] \\ B \arrow[r, "f"] & G_n \end{tikzcd}$$

where

$$E(f^*(\gamma^n)) = \{ (b, \vec{v}) \mid \vec{v} \in f(b) \}.$$

We clearly have a cover $\hat{f}$ making f a bundle map from $f^*(\gamma^n) \to \gamma^n$.

Now (similar to the case for real bundles)

Theorem. Every complex n-plane bundle (over a paracompact base) admits a bundle map into γ^n.

* two ~~such~~ bundles are isomorphic $\Leftrightarrow$ their bases admit homotopic maps into G^n which are covered by bundle maps

* every bundle is isomorphic to a bundle induced by a map of its base to G_n

We are now interested in the product theorem for Chern classes of complex bundles over the same paracompact base B. We want

$$c(\omega \oplus \varphi) = c(\omega)c(\varphi).$$

We start with a lemma.

Lemma. $\exists!$ a polynomial with integer coeffs.

$$p_{m,n}(c_1, \ldots, c_m; c_1', \ldots, c_n')$$

so that

$$c(\omega \oplus \varphi) = p_{m,n}(c_1(\omega), \ldots, c_m(\omega); c_1(\varphi), \ldots, c_n(\varphi))$$

for every complex m- and n-plane bundles ω and φ over a common (paracompact) B.

Proof. We start by constructing a model for the situation: "γ_1^m" and "γ_2^n" over $G_m \times G_n$.

To define the bundles, we let

$$\pi_1 : G_m \times G_n \to G_m, \quad \pi_2 : G_m \times G_n \to G_n$$

and set

$$\gamma_1^m = \pi_1^*(\gamma^m), \quad \gamma_2^n = \pi_2^*(\gamma^n)$$

(What does this mean? In this specific case, it means

$$E(\gamma_1^m) = E(\gamma^m) \times G_n$$

$$\pi(\gamma_1^m) = \pi(\gamma^m) \times \mathrm{Id}$$

$$F(\gamma_1^m) = F(\gamma^m)$$

and so forth. But I see no reason to believe that there's a general pullback for vector bundles. Am I missing something?)

We then observe that

$$\gamma_1^m \oplus \gamma_2^n \cong \gamma^m \times \gamma^n$$

by construction.

Now recall that we have

$$\begin{aligned} a \times b &= (a \times 1) \cup (1 \times b) \\ &= \pi_1^* a \cup \pi_2^* b \end{aligned}$$

and that by the Kunneth formula, since $H^*(G_m)$ is torsion-free, we have

$$\times : H^*(G_m) \otimes H^*(G_n) \to H^*(G_m \times G_n)$$

an isomorphism. (even in $\mathbb{Z}$ coefficients).

Now that means $H^*(G_m \times G_n)$ is a polynomial ring on $c_i(\gamma_1^m)$ and $c_j(\gamma_2^n)$.

But clearly π_1 and π_2 are covered by bundle maps

$$\pi_1 : \gamma_1^m \to \gamma^m, \quad \pi_2 : \gamma_2^n \to \gamma^n,$$

so

$$c_i(\gamma_1^m) = \pi_1^* c_i(\gamma^m)$$

$$c_j(\gamma_2^n) = \pi_2^* c_j(\gamma^n)$$

by naturality of Chern classes. So whatever $c(\gamma_1^m \otimes \gamma_2^n)$ is, it is expressed uniquely as a polynomial in $c_i(\gamma_1^m)$ and $c_j(\gamma_2^n)$ since these are algebraically independent of one another by our last proposition & the Kunneth argument above.

This gives us $p_{m,n}$ for an example. How do we know this polynomial is universal?

We know from before that $\exists$ maps $f: B \to G_m$ and $g: B \to G_n$ so that

$$\omega \cong f^*(\gamma^m) \,, \quad \varphi \cong g^*(\gamma^n)$$

If $h: B \to G_m \times G_n$ is given by $f \times g$, then

$$\begin{tikzcd}
 & B \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
G_m & G_m \times G_n \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & G_n
\end{tikzcd}$$

commutes, ~~so~~ and $h^*(\gamma_1^m) \cong \omega$, $h^*(\gamma_2^m) \cong \varphi$.

Now since all these maps are covered by bundle maps, we have

$$\begin{aligned}
c(\omega \oplus \varphi) &= h^* c(\gamma_1^m \oplus \gamma_2^n) \\
&= p_{m,n}\big(h^* c_1(\gamma_1^m), \ldots, h^* c_m(\gamma_1^m), \\
&\qquad h^* c_1(\gamma_2^n), \ldots, h^* c_n(\gamma_2^n)\big) \\
&= p_{m,n}\big(c_1(\omega), \ldots, c_m(\omega); c_1(\varphi), \ldots, c_n(\varphi)\big).
\end{aligned}$$

$\square$

We now actually compute the polynomial $p_{m,n}$ by induction on $m+n$. Suppose

$$c(\gamma_1^{m-1} \oplus \gamma_2^n) = (1 + c_1(\gamma_1^{m-1}) + \dots + c_{m-1}(\gamma_1^{m-1}))(1 + \dots + c_n(\gamma_2^n))$$

Now consider $(\gamma_1^{m-1} \oplus \varepsilon^1)$ and γ_2^n over $G_{m-1} \times G_n$.

We know

$$c((\gamma_1^{m-1} \oplus \varepsilon^1) \oplus \gamma_2^n) = p_{m,n}\big(c_1(\gamma_1^{m-1} \oplus \varepsilon^1), \dots, c_m(\gamma^{m-1} \oplus \varepsilon^1);$$
$$c_1(\gamma_2^n), \dots, c_n(\gamma_2^n)\big)$$

$$= p_{m,n}(c_1(\gamma_1^{m-1}), \dots, c_{m-1}(\gamma_1^{m-1}), 0; c_1(\gamma_2^n), \dots, c_n(\gamma_2^n))$$

and by inductive hypothesis

$$= c(\gamma_1^{m-1} \oplus \gamma_2^n) = (1 + c_1(\gamma_1^{m-1}) + \dots + c_{m-1}(\gamma^{m-1}))(1 + \dots + c_n(\gamma_2^n))$$

So if $c_i = c_i(\gamma_1^{m-1})$ and $c_j' = c_j(\gamma_2^n)$,

$$p_{m,n}(c_1, \dots, c_{m-1}, 0; c_1', \dots, c_n') = (1 + \dots + c_{m-1})(1 + \dots c_n').$$

an unknown

Introduce c_m and let

$$p_{m,n}(c_1,\dots,c_m;c_1',\dots,c_n') = (1+c_1+\dots+c_m)(1+\dots+c_n) \bmod c_m$$

in $\mathbb{Z}[c_1,\dots,c_m,c_1',\dots,c_n']$. Similarly, congruent ~~md~~ mod c_n', and so congruent mod $c_m c_n'$.