Complex Bundles and the Chern Class

Our strategy now is to impose some additional structure on the situation in the hopes of getting more results.

Definition. A <u>complex</u> <u>vector</u> <u>bundle</u> of (complex) dimension n over B is a space $\pi: E \to B$ so that each $x \in B$ has a neighborhood U so that

- $\pi^{-1}(x)$ has a complex vector space structure
- $\pi^{-1}(U) \cong U \times \mathbb{C}^n$ under a map that's a homeomorphism <u>and</u> complex linear from $b \times \mathbb{C}^n \to \pi^{-1}(b)$ for each $b \in U$.

We define the various other bundle constructions (Whitney sum, tensor product, bundle constructed from functor) as before.

It is clear that a complex vector bundle of dimension n is, in particular, a real $2n$-bundle. Can we ~~go~~ work from a real $2n$-bundle and promote it to a $\mathbb{C}$ bundle?

<u>Definition</u>. A complex structure on a $2n$-plane bundle ξ is a mapping

$$J : E \to E$$

which is $(\mathbb{R})$-linear on each fiber and satisfies $J(J(v)) = -v$.

<u>Lemma</u>. Given such a J, the scalar multiplication

$$(x + iy)\, v = xv + J(yv)$$

turns each fiber into a complex vector space.

You can also define a complex structure on M^{2n} by taking charts in $\mathbb{C}^n$ and checking that the transfer maps from chart to chart are holomorphic.

Lemma. If ω is a complex n-plane bundle, the underlying real 2n-plane bundle $\omega_{\mathbb{R}}$ has a preferred orientation.

Proof. Choose a (complex) basis $v_1, \ldots, v_n$ for a fiber. Now consider the real basis

$$a_1, ia_1, \ldots, a_n, ia_n$$

for the fiber. This basis has an orientation. Now observe that $GL(n, \mathbb{C})$ is connected, so we may deform $v_1, \ldots, v_n$ to any other basis $v_1', \ldots, v_n'$ for the space.

and

- Each $x \in M$ has an open neighborhood $\mathcal{U}$ with a diffeomorphism $h: \mathcal{U} \to \mathcal{U}' \subset \mathbb{C}^n$ so that dh is complex linear.

We also define

Definition. A smooth mapping $f: M \to N$ between complex manifolds is ~~tr~~ holomorphic $\Leftrightarrow$ Df is complex linear.

We note that this condition can be written differentially. If J is a smooth almost-complex structure on M, J is a complex structure $\Leftrightarrow$

$$[Jv, Jw] = J[v, Jw] + J[Jv, w] + [v, w].$$

Where $[\,,\,]$ is the Lie bracket.

Example. Suppose $U \subset \mathbb{C}^n$. We define $TU = U \times \mathbb{C}^n$ to have the complex structure

$$J(u,v) = (u, iv).$$

If $f : U \to U' \subset \mathbb{C}^p$, we have an induced ($\mathbb{R}$) linear differential $Df_x : T_x U \to T_{f(x)} U'$.

If Df is complex-linear / commutes with J, then we say f is holomorphic or complex analytic.

Now a manifold whose tangent bundle has a complex structure is said to have an almost complex structure.

Definition. A real $2n$-manifold has a complex structure if

- it has an almost complex structure (that is, TM has a complex structure as a vector bundle)

Proof. We must check

$$\big((x+iy)(a+ib)\big)v = (x+iy)\big((a+ib)v\big)$$

On the left, we get

$$\big((ax-by)+i(ay+bx)\big)\vec{v} = (ax-by)\vec{v} + J\big((ay+bx)\vec{v}\big).$$

On the right,

$$(x+iy)\big(a\vec{v} + J(b\vec{v})\big) = xa\vec{v} + J(bx\vec{v}) + J\big(ay\vec{v} + J(yb)\vec{v}\big)$$

$$= xa\vec{v} + J(bx\vec{v}) + J(ay\vec{v}) - yb\vec{v}. \square$$

Now we usually get complex vector bundles as the tangent bundles to complex manifolds. To see how they work, lets start with a complex chart.

It defines a complex-valued inner product on the fibers of the bundle so that

$$\langle v, w \rangle = \frac{1}{2}\left(|v+w|^2 - |v|^2 - |w|^2\right) + \frac{1}{2}i\left(|v+iw|^2 - |v|^2 - |iw|^2\right).$$

This inner product is almost **bilinear**: we have

$$\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$$

but

$$\langle v, \lambda w \rangle = \bar{\lambda} \langle v, w \rangle$$

Thus $\langle v, w \rangle = \overline{\langle w, v \rangle}$. We note that still

$$\langle v, v \rangle = |v|^2.$$

We will consider vectors orthogonal if $\langle v, w \rangle = 0$ (note that this is actually two real conditions).

such a deformation doesn't change orientation.

Thus the Euler class $e(\omega_R)$ is well defined for any complex n-plane bundle.

In fact

$$e(\omega_R \oplus \omega'_R) = e(\omega_R)\, e(\omega'_R).$$

We just need to check that

$$(\omega \oplus \omega')_R \underset{\substack{\text{as oriented} \\ \text{bundle}}}{\text{isomorphic}} \omega_R \oplus \omega'_R.$$

Definition. (Hermitian metric)

A Hermitian metric is a Euclidean metric on the underlying real bundle so that

$$|\vec{v}| = |i\vec{v}|$$

We need one last preliminary. Given

$$\begin{array}{ccc} E & & E_0 \\ \downarrow \pi & \text{there exists} & \downarrow \pi_0 \\ B & & B \end{array}$$

Theorem. To any oriented n-plane bundle ξ, there is an exact sequence called the <u>Gysin sequence</u>

$$\longrightarrow H^i(B) \xrightarrow{\cup e} H^{i+n}(B) \xrightarrow{\pi_0^*} H^{i+n}(E_0) \longrightarrow H^{i+1}(B) \rightarrow$$

in integer coefficients.

Proof. Start with the exact sequence of the pair

$$\begin{tikzcd}
\rightarrow H^j(E,E_0) \arrow[r] & H^j(E) \arrow[r] & H^j(E_0) \arrow[r, "\delta"] & H^{j+1}(E,E_0) \rightarrow \\
H^{j-n}(E) \arrow[u, leftrightarrow, "\cup u \ \text{(Thom isomorphism)}"'] \arrow[ur]
\end{tikzcd}$$

this map is $x \mapsto (x \cup u)|_E = x \cup (u|_E)$.

~~But $(u|_E) = e$.~~ And E deformation retracts to B, while under that cohomology isomorphism induced by π^* $(u|_E) \to e$.

So we can write this as

$$\rightarrow H^{j-n}(B) \xrightarrow{\cup e} H^j(B) \xrightarrow{\pi_0^*} H^j(E_0) \longrightarrow H^{j-n+1}(B)$$

this map is interesting. It's δ composed with inverse of Thom isomorphism.

$\square$

Now we can begin an inductive definition of Chern classes. We start with a complex bundle ω

$$\begin{array}{ccc} E(\omega_0) & & \\ \downarrow \pi_0 & & \\ E_0 & \subset & E \\ & & \downarrow \pi \\ & & B \end{array}$$

and build a canonical bundle ω_0 over the complement E_0 of the 0-section in $E(\omega)$.

Given $(x,v) \in E_0$, where $v \in \pi^{-1}(x)$, if $\exists$ a Hermitian metric on ω we can take $v^\perp \subset \pi^{-1}(x)$ (in the complex sense) to get a complex $(n-1)$-plane ~~bundle over~~ inside the fiber $\pi^{-1}(x)$. These planes define the new bundle as ~~a sub-bundle~~ something like a subbundle of the old one.

if we don't have a Hermitian metric, we can just define the fiber of ω_0 to be ~~π^{-1}~~ the quotient of the fiber $\pi^{-1}(x) / \text{complex subspace spanned by } \vec{v}$.

Now we have the Gysin sequence

$$\to H^{i-2n}(B) \xrightarrow{\cup e} H^i(B) \xrightarrow{\pi_0^*} H^i(E_0) \to H^{i-2n+1}(B) \to$$

for $i < 2n-1$, this gives us an isomorphism

$$H^i(B) \cong H^i(E_0)$$

Definition. The Chern classes of ω, $c_i(\omega) \in H^{2i}(B;\mathbb{Z})$ are defined by induction on the complex dimension on ω.

$$c_n(\omega) = e(\omega_{\mathbb{R}}), \in \text{the Euler class}$$

$$c_i(\omega) = (\pi_0^*)^{-1} c_i(\omega_0), \quad i < n$$

We set $c_i(\omega) = 0$ for $i > n$.

Check: Does this make sense?

We need to check,

* (π_0^*) is an isomorphism

  For $i<n$, $H^{2i}(B) \xrightarrow{\pi_0^*} H^{2i}(E_0)$
  is an isomorphism by above.

* Now for $i = n-1$, we observe
  that $c_{n-1}(\omega) = (\pi_0^*)^{-1} c_{n-1}(\omega_0)$

$$= (\pi_0^*)^{-1} e(\omega_0).$$

And c_{n-2}? Well that's

$$(\pi_0^*)^{-1} (\pi_0^*)^{-1} e((\omega_0)_0)$$

and then it's turtles all the way down until we reach dimension zero.

As usual, we let the total Chern class be

$$c(\omega) = 1 + c_1(\omega) + c_2(\omega) + \dots + c_n(\omega)$$

We claim that we can inductively invert

$$c(\omega)^{-1} = 1 - c_1(\omega) + \left(c_1(\omega)^2 - c_2(\omega)\right) + \dots$$

~~Proof. We have~~

$$\cancel{c(\omega)\,c(\omega)^{-1} = 1 + c_1(\omega) + c_2(\omega) + \dots}$$

$$\cancel{- c_1(\omega) - c_1(\omega)^2 - c_1(\omega)c_2(\omega) + \dots}$$

$$\cancel{+ c_1(\omega)^2 + c_1(\omega)^3 + c_1^2(\omega)c_2(\omega)}$$

$$\cancel{- c_2(\omega) - c_1(\omega)c_2(\omega) - c_2^2(\omega) -}$$

~~= I don't really see this.~~

as we did for the total SW class.

We now show

Lemma. The Chern classes are natural.

Proof. Suppose $f: B \to B'$ is covered by a bundle map from ω to ω'. We claim $c_i(\omega) = f^* c_i(\omega')$.

~~The~~ We proceed by induction on n. For $i = n$, we know c_n is natural because it is an Euler class. ~~Now for general i, we know that~~ This proves that Chern classes are natural for 1-plane bundles.

Now suppose ω is an n-plane bundle. Since ω_0 is an $n-1$ plane bundle, we

Know that Chern classes are natural on ω_0. But our bundle map

$$\begin{tikzcd} E \arrow[r, "\hat{f}"] \arrow[d] & E' \arrow[d] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

gives rise to a map between the ω_0, ω_0' bundles

$$\begin{tikzcd} E(\omega_0) \arrow[r, "\hat{\hat{f}}"] \arrow[d, "\pi"'] & E(\omega_0') \arrow[d, "\pi"] \\ E_0 \arrow[r, "\hat{f}"] & E_0' \end{tikzcd}$$

which is covered by an induced map $\hat{\hat{f}}$. Now this means that the Chern classes of ω_0 are natural (by induction) and so pull back to E_0 under $\hat{f}$.

but

$$\begin{tikzcd} E_0(\omega) \arrow[r, "\hat{f}"] \arrow[d, "\pi_0"] & E_0(\omega') \arrow[d, "\pi_0"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

commutes, so this means that the Chern classes of B' pull back to those of B, as desired. $\square$.

Lemma. If ε^k is the trivial complex k-plane bundle over B, then $c(\omega \oplus \varepsilon^k) = c(\omega)$

Proof. Consider $k=1$, and let $\varphi = \omega \oplus \varepsilon^1$. Now φ has a nonzero cross section, so

$$c_{n+1}(\varphi) = e(\varphi_R) = 0.$$

Thus $c_{n+1}(\varphi) = c_{n+1}(\omega)$.

Now let s be the cross-section. We have

$$\begin{tikzcd}
E(\omega) \arrow[d, "\pi"'] & & E(\varphi_0) \arrow[d, "\pi"] \\
B \arrow[rr, "s"] & & E_0(\underset{\varphi}{\underbrace{\omega \oplus \varepsilon^1}})
\end{tikzcd}$$

cross-section

We observe that (locally) we have

$$x \overset{s}{\longmapsto} (x, (0,1))$$

and this is covered by

$$(x,v) \overset{\hat{s}}{\longmapsto} ((x,(0,1)), (v,0))$$

Since the fiber of $E(\varphi_0)$ consists of everything in a fiber of $\omega \oplus \varepsilon^1$ which is $\perp$ to the given vector. So we have

$$\begin{tikzcd}
E(\omega) \arrow[r, "\hat{s}"] \arrow[d] & E(\varphi_0) \arrow[d] \\
B \arrow[r, "s"] & E_0(\omega \oplus \varepsilon^1 = \varphi)
\end{tikzcd}$$

Now by naturality, this means that

$$s^* c_i(\varphi_0) = c_i(\omega)$$

But by definition,

$$c_i(\varphi) = (\pi_0^*)^{-1} c_i(\varphi_0)$$

So

$$\pi_0^* c_i(\varphi_0) = c_i(\varphi_0)$$

So

$$\cancel{\pi}\; s^*(\pi_0^* c_i(\varphi)) = c_i(\omega).$$

But φ is a bundle over B, so if we think of the cross-section as a map into $E(\varphi)$, we must have $\pi(s(x)) = x$.

$$\begin{tikzcd}
E(\varphi) \arrow[r, hook] \arrow[d, shift left, "\pi"] & E_0(\varphi) \arrow[dl, shift left, "\pi_0"] \\
B \arrow[u, bend left, "s"] \arrow[ur, shift left, "s"] &
\end{tikzcd}$$

The same is true for s as a map into $E_0(\varphi)$, So we must have

$$s^* \circ \pi_0^* = \mathrm{Id}$$

and

$$c_i(\varphi) = c_i(\omega) \quad \text{as claimed} \qquad \square$$

We now define

Definition. The complex Grassmannian manifold $G_n(\mathbb{C}^{n+k})$ is the set of all complex n-planes through 0 in $\mathbb{C}^{n+k}$.

As before,

* $G_n(\mathbb{C}^{n+k})$ is a complex manifold with complex dimension nk
* $\exists\ \gamma^n(\mathbb{C}^{n+k})$, an n-plane bundle over $G_n(\mathbb{C}^{n+k})$ given by pairs (X, v) where $X \in G_n(\mathbb{C}^{n+k})$ and $v \in X$ is a vector in $\mathbb{C}^{n+k}$.