The Carpenter's Ruler Theorem II.

We now show

Theorem (Connelly, Demaine, Rote 2003)
The configuration space of embedded planar polygons w/fixed edgelengths has 2 path connected components, (clockwise and counterclockwise)

Actually, their theorem is stronger, as we'll see along the way.

Definition. A vectorfield $\vec{v}_i$ on the vertices $\vec{p}_i$ of a polygon is <u>expansive</u> if
$$\langle \vec{v}_i - \vec{v}_j, \vec{p}_i - \vec{p}_j \rangle > 0$$
for all i, j.

Now valleys (corresponding to outward pushing stresses) may be quite common - as many struts are incident to any given vertex.

But mountains (inward pulling stresses) are quite rare - they must correspond to bars in the original linkage, and hence either $\underline{0}$ or $\underline{2}$ are present.

However, any plane polygon P' has at least $\underline{3}$ convex vertices.

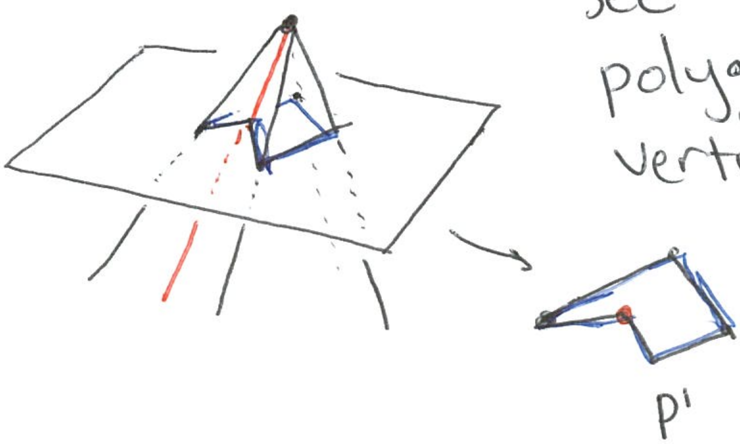
Of course, this leaves open the possibility that the set of vertices with $z = z_{max}$ is larger.

"all bar" framework is <u>definitely</u> rigid, the planar tensegrity is infinitesimally rigid <=> it has an equilibrium stress, nonzero on each strut.

~~So now we are going to prove this framework has no equilibrium stress except the zero stress.~~

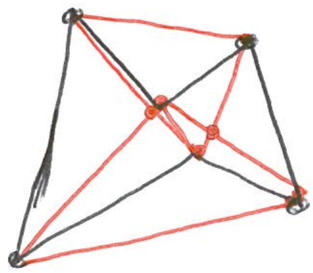
If so, $\exists$ a Maxwell-Cremona lift of the framework, with exterior vertices at height zero, and all struts strictly valley.

Motivating Case. Suppose a single vertex is at maximum z height. Slicing, we see a little embedded planar polygon where convex vertices are mountains and reflex vert are valleys.



We claim that any configuration of a polygon which is not already convex has an expansive motion.

First, if any pair of edges is at angle π, we can just treat them as a single edge. Now add struts between every pair



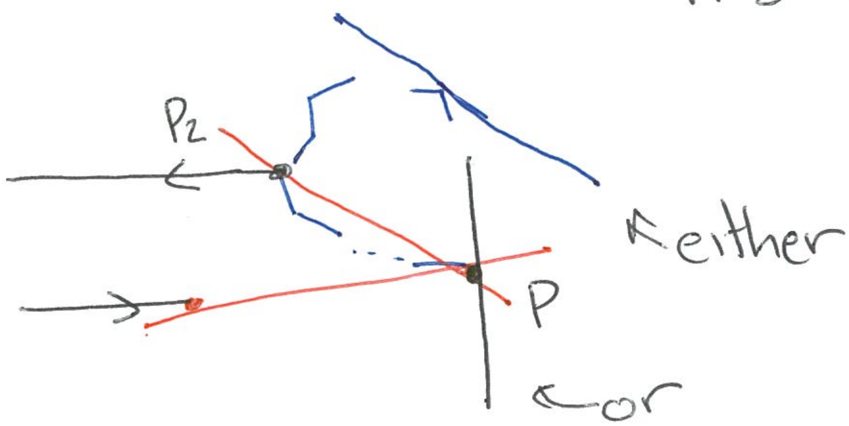
of remaining vertices, and add new vertices at crossings to construct a planar tensegrity.

Claim. This tensegrity is infinitesimally flexible

We proceed in stages. First, add enough additional struts between new verts to triangulate the planar tensegrity.

By Roth & Whitely's theorem, since the

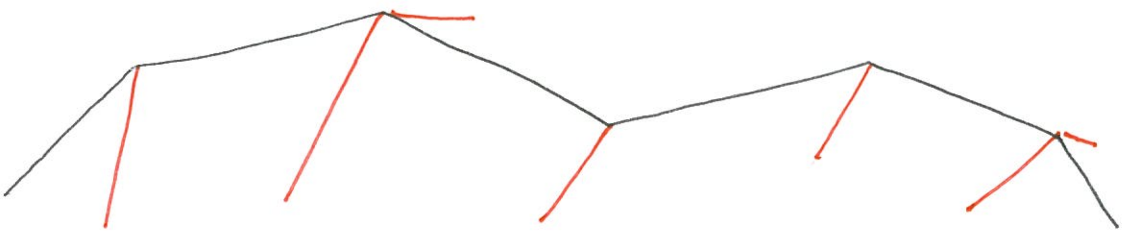
Join this point p to ~~p~~ p_1 and p_2, and assum
it's not p_2.



If the arc joining p to p_2 ever ventures ~~above~~ to the side opposite p_1 there's a point at maximum distance, ~~tha~~ which makes another left turn.

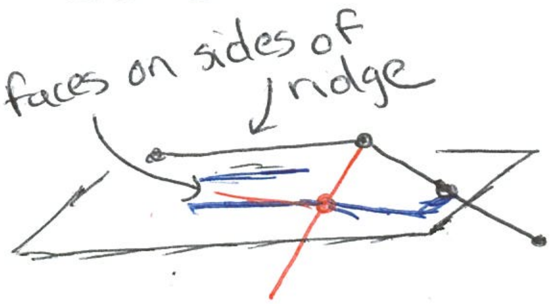
If not, p_2 itself is a left turn! □

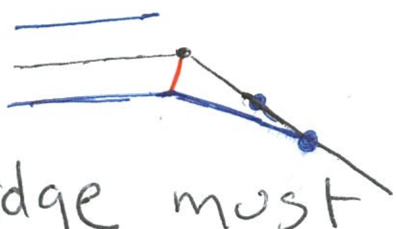
Using the lemma, we have a contradictio
- each convex vertex is mountain, and there is only <u>one</u> mountain edge connected to vertex v left to use.



No edge between vertices at maximum height is a strict valley. Therefore, every such edge is a bar of the original polygon, and a (connected component) is an open polygon.

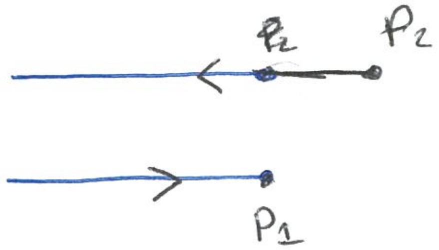
Consider an endpoint v. Repeating the slice-just-below trick we see the intersection of the faces on





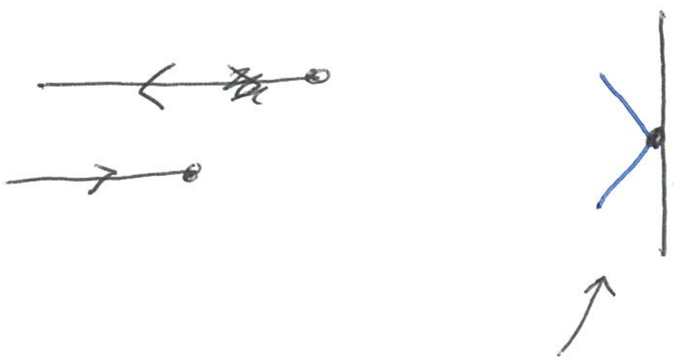
each side of the ridge must

intersect the horizontal plane in lines parallel to the ridge



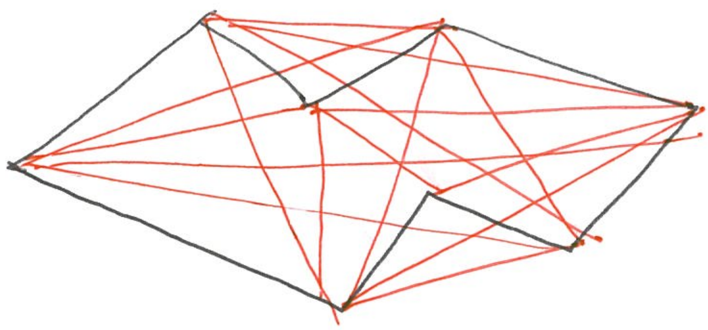
Lemma. There is no way to join points on parallel lines, (by an embedded polygonal arc) without at least 2 convex (or left turning) vertices.

Proof sketch.



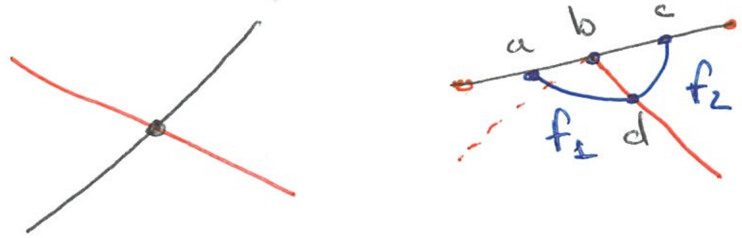
Some point is rightmost and this is one left turn already (if >1 point is rightmost, we're already done).

Thus the maximum-z set is the entire original polygon.



~~We et~~
~~Claim. This~~

Suppose the polygon is not convex. Then some strut crosses a bar as

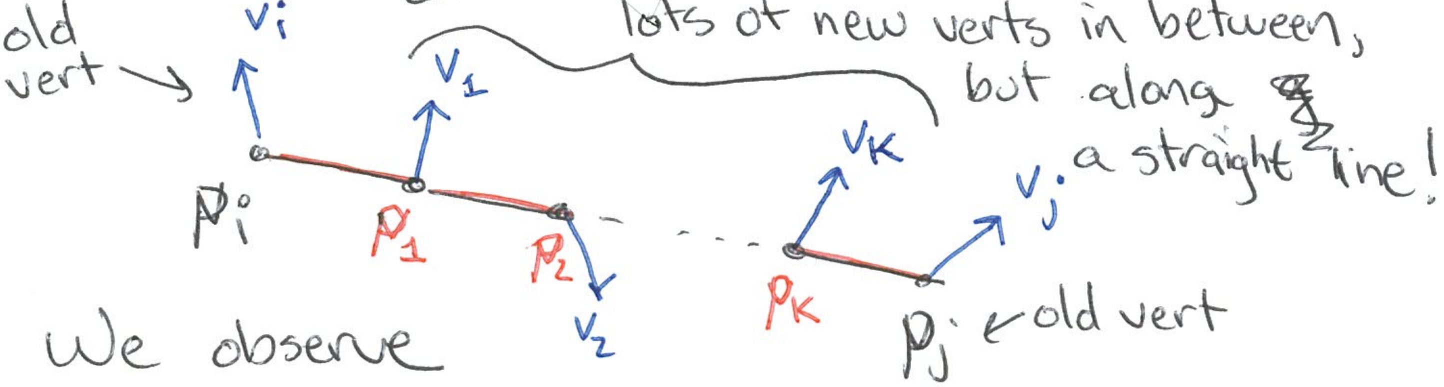


The strut <u>cannot</u> be strictly valley- as abc are colinear in the lift b/c the whole edge is horizontal at (max) height 0, the faces f_1 and f_2 are coplanar. XX

This gives us a contradiction which allows us to conclude that

~~in~~ polygon not convex $\Rightarrow$ augmented tensegrity is infinitesimally flexible.

We now argue that this implies that the original polygon has an ~~infin~~ expansive vectorfield. First, our vector field is going to be the restriction of the infinitesimal flex to the ~~old~~ old vertices. So consider



We observe

$$\langle v_i - v_j, p_i - p_j \rangle = \langle v_i - v_1 + v_2 - v_2 + v_2 - \dots - v_K + v_K - v_j, \; p_i - p_j \rangle$$

$$= \langle v_i - v_1, p_i - p_j \rangle + \langle v_1 - v_2, p_i - p_j \rangle$$

$$+ \dots + \langle v_k - v_j, p_i - p_j \rangle$$

But $p_i - p_j = \lambda_1 (p_i - p_1) = \lambda_2 (p_1 - p_2) = \dots$

$$= \lambda_k (p_{k-1} - p_k) = \lambda_j (p_k - p_j)$$

where all the λ are positive, so we have

$$= \lambda_1 \langle v_i - v_1, p_i - p_1 \rangle + \dots + \lambda_j \langle v_k - v_j, p_k - p_j \rangle$$

$=$ sum of positive stuff $=$ positive.

From here, ODE stuff.

reach the desired configuration. The fourth property prevents multiple components from flying apart from each other so quickly that they never actually straighten or convexify.

We begin in Section 2 with background and definitions. Then in Section 3 we define the precise constraints we need of an energy function and give examples of such energy functions. Section 4 establishes the main mathematical result, that gradient flow produces the desired smooth motion. Section 5 describes the algorithm to find an exact piecewise-linear motion and proves that its running time is finite. Section 6 gives explicit bounds on the running time in terms of n and geometric features of the input. Section 7 describes experiments with an implementation of our approach, and shows the resulting animations and running times. We conclude in Section 8

2 Background: Arc-and-Cycle Sets

We now define the objects of interest. An *arc-and-cycle set* A is a finite collection of planar polygonal arcs and polygonal closed curves. A *configuration* $V = [v_1, v_2, \ldots]$ of A is an assignment of coordinates to vertices such that the edge lengths match those in A. If A has n vertices, the *configuration space* of A, denoted $X(A)$, can be viewed as the algebraic subvariety of $\mathbb{R}^{2n}$ determined by fixing the length of each edge. The *embedded* configurations of A—configurations without self-crossing—are denoted $EX(A)$.

A configuration of an arc-and-cycle set is *outer-convex* if each outermost connected-component of A is either straight (when it is an arc) or convex (when it is a cycle). A motion of a configuration is *strictly expansive* if it does not decrease any vertex-to-vertex distance, and strictly increases all of the vertex-to-vertex distances between pairs of vertices that are not forced to have constant distance because they are connected by a straight chain of edges or because they are on or inside a common convex cycle. A motion is merely *expansive* when it does not decrease any vertex-vertex distance, and increases at least one such distance.

The main result of [9] establishes the existence of such motions, which we use extensively:

THEOREM 2.1. *Any arc-and-cycle set admits a strictly expansive motion until it is outer-convex.*

3 Energy Functions

Next we consider energy functions whose minimization forces the linkage to "repel itself". The gradient of any such function will then define a motion of the linkage towards an outer-convex configuration that avoids crossings as desired.

3.1 Definition and Required Properties. An *energy function* is a function from embedded configurations $EX(A)$ to the nonnegative real numbers $\mathbb{R}^+$. We call an energy function *admissible* if it has four properties defined below: it must be C^2, charge, repulsive, and separable. (We can define a version of admissibility for $C^{1,1}$ functions instead of C^2, but it is much harder to work with.)

3.1.1 Charge. An energy function E is *charge* if it approaches $+\infty$ on the boundary of $EX(A)$, that is, if it becomes infinite as the linkage approaches any self-crossing configuration.

This requirement is an adaptation of an idea from the literature of knot energies (cf. [10]) to capture the idea that our energy functional must avoid self-crossing configurations. The inspiration for the name "charge" comes from electrostatics, where it takes an infinite amount of work to pull a pair of point charges together until they coincide.

3.1.2 Repulsive. An energy function E is *repulsive* if it decreases to first order under any strictly expansive motion of A.

This requirement captures the idea that the vertices and edges of the linkage should roughly repel each other under the gradient flow of the energy.

3.1.3 Separable. For an arc-and-cycle set A with connected components $A_1, \ldots, A_n$, an energy function E is *separable* if it can be written in the form

$$E(A) = \sum_{i,j=1}^{n} E_{ij}(A_i, A_j), \tag{3.1}$$

where each *two-component energy* E_{ij} is an energy function on the arc-and-cycle set $A_i \cup A_j$ that itself is C^2, repulsive, and charge; and furthermore the contribution of E_{ij} to the gradient of E approaches zero as the distance between A_i and A_j grows.

This requirement enforces that, as connected components of A become far away from each another, the repulsion between them has little impact on the gradient of the energy.

3.2 Example. We now give an example of an energy function that obeys our criteria. The basic idea is to sum powers of reciprocals of distances between vertices and edges of the arc-and-cycle set. This idea immediately leads to the charge property: as a distance approaches zero, the reciprocal approaches $+\infty$. We use a particular definition of distance between a vertex and edge so that the energy function is C^∞.

Specifically, the *elliptic-distance energy* of an arc-and-cycle set A with vertex set V and edge set E is defined by

$$E(A) := \sum_{\substack{\text{edge } \{v,w\} \\ \text{vertex } u \notin \{v,w\}}} \frac{1}{(\|u-v\|+\|u-w\|-\|v-w\|)^2}. \tag{3.2}$$

where the denominator is the squared *elliptic distance* between vertex u and edge $\{v, w\}$. For any edge $\{v, w\}$, the level sets of the summand in the elliptic-distance energy, as we vary the position of vertex u, are a family of ellipses with foci at v and w which converge at zero to the edge $\{v_i, v_j\}$.

PROPOSITION 3.1. *Elliptic-distance energy is admissible.*

Proof. This energy is C^∞ on the interior of $EX(A)$ and is therefore also C^2. Because the denominator of the summand vanishes precisely when vertex u is on the edge (v, w), the energy is charge. Any expansive motion cannot increase any of the summands, and it must increase a positive term in at least one of the denominators, while leaving all negated terms alone. Thus the energy is repulsive. Finally, because we can split the sum up according to which connected-component of A the edge (v, w) and the vertex u belong to, while the derivative of the summand approaches zero as the distances $\|u - v\|$ and $\|u - w\|$ become large, the energy is separable. □

4 Gradient Flow Almost Unfolds Linkages

This section proves our main mathematical result: for any $\epsilon > 0$, the negative gradient flow of any admissible energy functional moves any linkage configuration to within distance ϵ of an outer-convex configuration in finite time.

4.1 Existence of Gradient Flow. We first observe that the gradient flow is well-defined:

PROPOSITION 4.1. *Given any embedded arc-and-cycle set A, the downhill gradient flow $A(t)$ of A under any admissible energy function E exists for all time $t \geq 0$ and is as smooth (in t) as the energy function E (in space).*

Proof. Because energy only decreases under gradient flow, we can restrict to the closed subspace $EX^+(A)$ of $EX(A)$ where $E \leq E(A) + 1$. Because E is C^2, the integral curve $V(t)$ of $-\nabla E$ through A exists for all time, unless it approaches the boundary of this space. But energy approaches $+\infty$ along the boundary and energy strictly decreases along the path, so this cannot happen. □

4.2 Main Theorem. We now prove our main theorem:

THEOREM 4.1. *If A is an arc-and-cycle-set and E is an admissible energy function on $EX(A)$, then for any $\epsilon > 0$ the motion $A(t)$ defined by the downhill gradient flow of E carries $A(0)$ to within ϵ of an outer-convex configuration in finite time.*

Proof. A standard result in dynamical systems says that any trajectory of the negative gradient flow $A(t)$ either weakly converges to some configuration of A that is critical for E or $A(t)$ leaves any compact neighborhood of $A(0)$ in finite time.

Because E is repulsive, Theorem 2.1 implies that any critical configuration of A is outer-convex. So in the first case there is nothing more to prove.

We focus on the second case. We can split A into n sublinkages $A_i(t)$, so that the components of each A_i remain within a bounded distance of one another for all time. In this case $A_i(t)$ remains within a compact subspace of $EX(A_i)$. We define a compact subspace of this space by restricting our attention to the space $EX^+(A_i)$ of configurations with $E_{ii} \leq E(A(0)) + 1$. Here we have used separability of E to write $E(A) = \sum_{i,j} E_{ij}(A_i, A_j)$ where each E_{ij} is a C^2, repulsive, charge energy function on $EX(A_i \cup A_j)$.

Now removing an ϵ-neighborhood of the outer-convex configurations leaves a subspace S_i on which $\|\nabla E_{ii}\|$ is bounded below by some $G_i > 0$, because this removes a neighborhood of the critical configurations for E_{ii} (by Theorem 2.1 and because E_{ii} is repulsive).

Because the A_i are drifting further apart, and E is separable, for each E_{ij} there is some finite time after which each $\|\nabla E_{ij}\| < G_i/2n$. After this point, the gradient flow of E must reduce each E_{ii} at rate at least $G_i/2$. But each $E_{ii}(A_i(t))$ is finite at this point and must always be non-negative, so for all t greater than some t_i, $A_i(t)$ must be outside S_i.

By definition, the complement of S_i consists of configurations with $E_{ii} > E(A_i(0))$ and configurations within ϵ of an outer-convex configuration. But $E_{ii}(A_i(t_i)) < E(A_i(0))$, so we must be in the second case: $A_i(t)$ is close to an outer-convex configuration for $t > t_i$. So for any $t > \max_i t_i$, $A(t)$ is close to an outer-convex configuration, completing the proof. □

5 Algorithm

This section presents an algorithm for computing a piecewise-linear motion from an initial configuration to an outer-convex configuration. This path is computed by first selecting a particular admissible energy function, expressing the energy function in terms of a suitable parameterization, and then applying Euler integration along the downward gradient path to get a series of "snapshots" of our linkage with decreasing energy which can be joined by linear interpolation in our parameter space. The algorithm terminates when we are sufficiently close to an energy-critical configuration to complete the motion by linear interpolation. As shown in Section 4, any critical configuration is guaranteed to correspond to an outer-convex configuration as desired.

5.1 Parameterizing the Configuration Space of an Arc. We start by considering the case when A consists of a single arc of $n-1$ edges. Refer to Figure 2(a). Let $V = [v_1, v_2, \ldots, v_n]$ denote the positions of the n vertices