The Carpenter's Ruler Theorem.

Suppose we fix edgelengths (thus restricting to a slice of $G_2(\mathbb{R}^n)$ of codimension $(n-1)$) and then further remove configurations where edges cross each other (removing large, top-dimensional chunks). Is what's left connected? Contractible?

Q. Can every embedded polygon with fixed edgelengths be reconfigured through embedded polygons to a convex polygon?

We will need a few tools.

Definition. A tensegrity is an embedded planar graph G with each edge marked strut, cable, or bar.

The configuration space of a tensegrity consists of ~~configurations~~ embeddings of the same graph, G' where

length $e_i' \geq$ length e_i, if e_i is a strut
length $e_i' =$ length e_i, if e_i is a bar
length $e_i' \leq$ length e_i', if e_i is a cable.

Definition. A tensegrity G is rigid if the connected component of the configuration space containing G is a ~~point~~ copy of $E(2)$.

Infinitesimal rigidity is a stronger condition that's easier to detect.
An infinitesimal motion of a tensegrity is an assignment of "velocity" vectors v_i to the vertices p_i of G.

so that

$$\langle (v_i - v_j), p_i - p_j \rangle \geq 0, \text{ if } e_{ij} \text{ is a strut}$$

$$\langle v_i - v_j, p_i - p_j \rangle = 0, \text{ if } e_{ij} \text{ is a bar}$$

$$\langle v_i - v_j, p_i - p_j \rangle \leq 0, \text{ if } e_{ij} \text{ is a cable.}$$

Theorem. If there is a path of configurations of G starting at $G(0) = G$, then

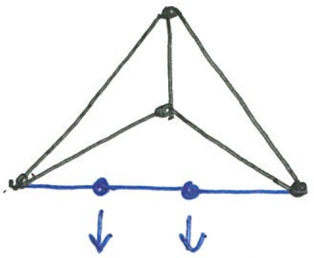
$$v_i = \frac{d}{dt} p_i(t) \Big|_{t=0}$$

is an infinitesimal motion. Thus,

(every infinitesimal motion is a tangent vector to a motion in $E(2)$) $\Rightarrow$ G is rigid.

"infinitesimal rigidity $\Rightarrow$ rigidity"

The converse is not true, alas.



This tensegrity is not infinitesimally rigid, but it is rigid. The vectors shown are an i.m.

We can formalize things a little more with the rigidity matrix

$$\begin{array}{c} \#\text{rows} \\ = \\ \#\text{edges} \end{array} \left\{ \underbrace{\begin{bmatrix} & & & \\ - & \vec{p}_i - \vec{p}_j & - & \vec{p}_j - \vec{p}_i \\ & \uparrow & & \uparrow \\ & i\text{th position} & & j\text{th position} \end{bmatrix}}_{\#\text{ columns} = 2 \cdot \#\text{ vertices}} \right. \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \\ \lambda_k \end{bmatrix}$$

derivatives of lengths

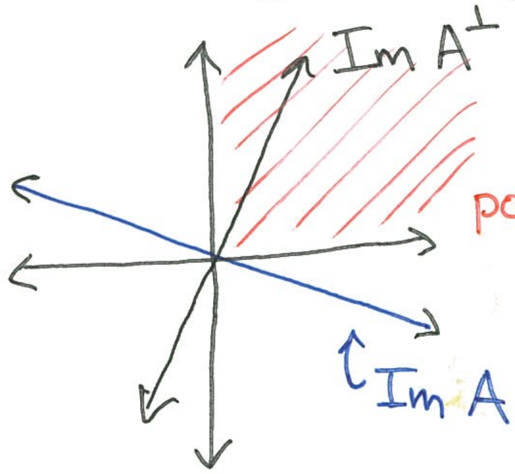
velocities

Now we see:

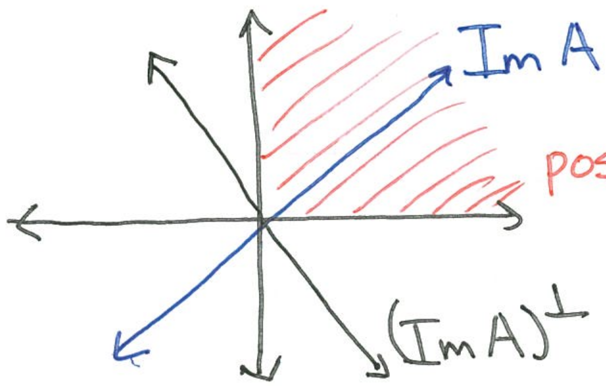
infinitesimal rigidity <=> rigidity matrix has rank $2|V|-3$

dimensional of $E(2)$

Here's a big idea: suppose we have a linear programming problem $Ax \geq 0$. Then



—or—



This is called the Farkas alternative theorem: ~~in any either Im~~

$$\cancel{\text{Im}(A) \text{ in } (\mathbb{R}^n)^{\geq 0} \Rightarrow (\text{Im } A)^\perp \cap (\mathbb{R}^n)^{>0} = \emptyset}$$

$$\text{Im } A \cap (\mathbb{R}^n)^{>0} \neq \emptyset \Rightarrow (\text{Im } A)^\perp \cap (\mathbb{R}^n)^{>0} = \emptyset$$

(and of course vice versa).

In our case, we like to use the version that Wikipedia calls Gordan's Theorem.

Either $Ax<0$ has a solution -or-

$A^T y = 0$ has a nonzero solution $y \geq 0$.

For tensegrities, the transpose of the rigidity matrix is the matrix

$$\begin{bmatrix} & | & \\ & p_i - p_j & \\ & | & \\ & p_j - p_i & \\ & | & \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} = \begin{bmatrix} \omega_1 = \sum_i (p_1 - p_i)\lambda_i \\ \vdots \\ \omega_n = \sum_i (p_n - p_i)\lambda_i \end{bmatrix}$$

"~~Forces~~ stress" on each strut, bar, cable

"net force" at each vertex.

This is the heart of the stress rigidity theorem.

~~There is an eq~~

Definition. We say a set of λ_{ij} is a <u>stress</u> if

$\lambda_{ij} \geq 0$, if edge e_{ij} is a strut

$\lambda_{ij} \leq 0$, if edge e_{ij} is a cable

(λ_{ij} has either sign if e_{ij} is a bar)

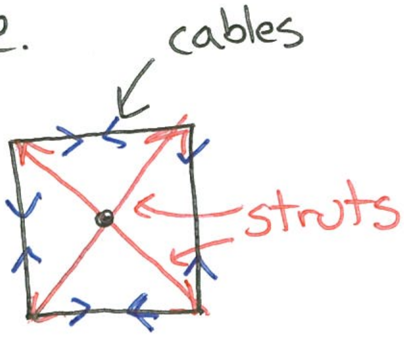
and an <u>equilibrium stress</u> if

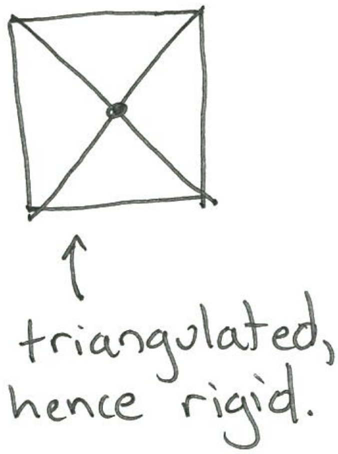
$$\sum_j \lambda_j (\vec{p}_i - \vec{p}_j) = \vec{0} \quad \text{for all } i \in 1, \dots, |V|.$$

Theorem. (Roth & Whitely, 1981)
A tensegrity is infinitesimally rigid <=>

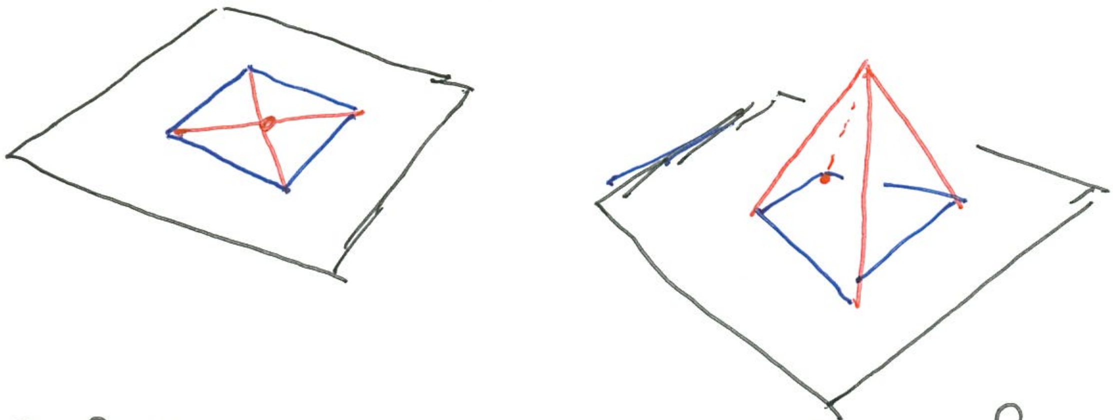
1. it has an equilibrium stress which is nonzero on each strut & cable
2. the corresponding "all bars" linkage is infinitesimally rigid.

Example.



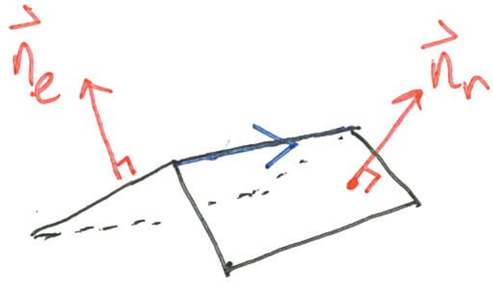


Now we get to something really cool - a connection between surface theory and tensegrity theory.



Definition. A polyhedral lifing of a tensegrity which is a planar graph is an assigment of z-heights ~~f~~ so that ~~ex~~ the vertices around each face lie in a plane, and the exterior face lies in the $z=0$ plane.

Given a lifting, at each edge there are two face normals. Assigning an orientation to the edge, there is a right normal $\vec{n}_r$ and a left normal $\vec{n}_\ell$.



We say the edge is a

mountain, if $\langle \vec{n}_r \times \vec{n}_\ell, \vec{e} \rangle > 0$

valley, if $\langle \vec{n}_r \times \vec{n}_\ell, \vec{e} \rangle < 0$.

flat, if $\langle \vec{n}_r \times \vec{n}_\ell, \vec{e} \rangle = 0$

Note that $\vec{n}_r \times \vec{n}_\ell$ ane $\vec{e}$ are colinear by construction, so the last condition could also be $\vec{n}_\ell = \vec{n}_r$.

Maxwell-Cremona Theorem. (1890, 1864, 1982)

A planar tensegrity has an equilibrium stress $\Leftrightarrow$ it has a polyhedral lifting.

Further, a lifting corresponds to a stress which is

positive ~~$\Leftrightarrow$~~ the edge is mountain

negative ~~$\Leftrightarrow$~~ the edge is valley

zero $\Leftrightarrow$ the edge is flat.