Variations and Curves.

Using our fundamental lemma:

$$\int_a^b |\vec{\alpha}(t)|\,dt \geq \left| \int_a^b \vec{\alpha}(t)\,dt \right|$$

for vector-valued functions, we can now prove.

Proposition. The shortest path between two points is a straight line.

Proof. Suppose we have any parametrized curve $\vec{\beta}(t)$ joining $\vec{\beta}(0)$ and $\vec{\beta}(1)$.

$$\text{length}(\vec{\beta}(t)) = \int_0^1 |\vec{\beta}'(t)|\,dt$$

$$\geq \left| \int_0^1 \vec{\beta}'(t)\,dt \right| = |\vec{\beta}(1) - \vec{\beta}(0)|.$$

This is very satisfying! But it's also very specialized.

Example. At ~~you~~ some point in your math education (high school? calc I?) you may have learned how to find the minimum of a quadratic function without calculus.

$$\min_x x^2 + bx + c = \min_x \left(x^2 + bx + \frac{b^2}{4}\right) + \left(c - \frac{b^2}{4}\right)$$

$$= \min_x (x + b/2)^2 + (c - b^2/4)$$

$$= c - b^2/4 \quad \text{at} \quad x = -b/2.$$

We can compare this to the

"calculus proof": If $f(x) = x^2 + bx + c$, then $f'(x) = 2x + b$, and $f'(x) = 0$ when $2x + b = 0$, or $x = -b/2$. At that point

$$f(-b/2) = \frac{b^2}{4} - \frac{b^2}{2} + c = c - \frac{b^2}{4}.$$

Since there is only one critical point and $f(x)$ is concave up, this critical point is the global min.

1. Take the derivative $f'(x)$.
2. Solve $f'(x) = 0$ to find critical point
3. Compute $f''(x)$ at critical point to identify local mins, maxes, and saddle points.

Proposition. If $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^2$ is in the form $\vec{\alpha}(t) = (t, y(t))$ then

$$\text{length } \vec{\alpha}(t) = \int_0^a \sqrt{1 + y'(t)^2}\, dt$$

Proof. Homework.

Definition. ~~We say~~ A <u>functional</u> $F(y)$ is a map (functions) $\to$ (numbers) in the form

$$F(y) = \int_0^a f(x, y(x), y'(x))\, dx$$

where $f : (0,a) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Example. length $\vec{\alpha}(t)$ is a functional of $y(t)$ with $f(t, y(t), y'(t)) = \sqrt{1 + y'(t)^2}$.

We would like to "differentiate $F(y)$ with respect to y", but what does that even mean?

Idea: F is a function of many variables, so we should take directional derivatives.

Definition. If $v(t)$ is a function, then

$$\delta F(y;v) = \lim_{h\to 0} \frac{F(y+hv)-F(y)}{h}$$

$$= \frac{d}{dh} F(y+hv)\Big|_{h=0}$$

is called the variation of F in direction v, at y.

Lemma. If y_* minimizes F over all functions $y : (0,a) \to \mathbb{R}$ with $y(0) = 0$ and $y(a) = b$, then

$$\delta F(y_*; v) = 0$$

for all $v : (0,a) \to \mathbb{R}$ with $v(0) = v(a) = 0$.

That is, "minimums are critical points".

Notice that the boundary conditions on the variation – $v(0) = v(a) = 0$ are different than the boundary conditions on y_* itself.

This is because $y_* + hv$ has to have $(y_* + hv)(0) = 0$ and $(y_* + hv)(a) = b$.

Now we want to compute the variation. So suppose

$$F(y) = \int_0^a f(x, y(x), y'(x))\, dx$$

Then

$$\delta F(y;v) = \frac{d}{dh}\left(\int_0^a f(x, y(x)+hv(x), y'(x)+hv'(x))\, dx \right)\Bigg|_{h=0}$$

$$= \int_0^a \frac{d}{dh} f(x, y(x)+hv(x), y'(x)+hv'(x))\Big|_{h=0}\, dx$$

$$= \int_0^a \frac{\partial}{\partial y} f(x, y(x), y'(x))\, v(x)\, dx + \int_0^a \frac{\partial}{\partial y'} f(x, y(x), y'(x))\, v'(x)\, dx.$$

Where by $\frac{\partial}{\partial y} f(x, y(x), y'(x))$ we mean "the partial ~~of~~ $\frac{\partial}{\partial b} f(a, b, c)$".

This is computed at a fixed value of x, so it's a standard partial.

Now consider

$$\int_0^a \left(\frac{\partial}{\partial y'} f(x,y(x),y'(x))\right) v'(x)\, dx.$$

Recalling ~~that the~~ integration by parts

$$\int_0^a u(x)\, v'(x)\, dx = u(x)v(x)\Big|_0^a - \int_0^a u'(x)v(x)\, dx$$

we see that

$$\int_0^a \left(\frac{\partial}{\partial y'} f(x,y(x),y'(x))\right) v'(x) =$$

$$\left(\frac{\partial}{\partial y'} f(x,y(x),y'(x))\right) v(x)\Big|_0^a -$$

$$\int_0^a \left[\frac{d}{dx}\left(\frac{\partial}{\partial y'} f(x,y(x),y'(x))\right)\right] v(x)\, dx.$$

You have to read these derivatives carefully: the $\partial/\partial y' f(x, y(x), y'(x))$ means $\frac{\partial}{\partial c} f(a,b,c)$. That is, the derivative with respect to the third argument.

The $\frac{d}{dx}\left(\frac{\partial}{\partial y'} f(x, y(x), y'(x))\right)$ means "the derivative with respect to x of the <u>entire</u> <u>expression</u>, computed via the chain rule".

This will hopefully be more clear when we do an example.

Now since $v(0) = v(a) = 0$,

$$\left(\frac{\partial}{\partial y'} f(x, y(x), y'(x))\right) v(x) \Big|_{x=0}^{a} = 0$$

and we have shown

$$\delta F(y; v) = \int_0^a \left(\frac{\partial}{\partial y} f(x, y(x), y'(x)) - \frac{d}{dx}\left(\frac{\partial}{\partial y'} f(x, y(x), y'(x))\right)\right) v(x)\, dx$$

Now for y to be a minimizer, this integral has to vanish for <u>all</u> $v(x)$ with $v(0) = v(a) = 0$. This is true $\Leftrightarrow$

$$\frac{\partial}{\partial y} f(x, y(x), y'(x)) = \frac{d}{dx}\left(\frac{\partial}{\partial y'} f(x, y(x), y'(x))\right)$$

for all x in $(0, a)$.

This is called an Euler-Lagrange equation.

Example. $F(y) = \int_0^a \sqrt{1+y'(x)^2}\, dx$.

$$\frac{\partial}{\partial y} \sqrt{1+(y')^2} = 0 \qquad (y \text{ does not appear})$$

$$\frac{\partial}{\partial y'} \sqrt{1+(y')^2} = \frac{\cancel{1}}{\cancel{2}}\left(1+(y')^2\right)^{-1/2} \cdot \cancel{2}y'$$

$$= \frac{y'}{\sqrt{1+(y')^2}}$$

So the E-L equation is simply

$$\frac{d}{dx} \frac{y'(x)}{\sqrt{1+(y'(x))^2}} = 0.$$

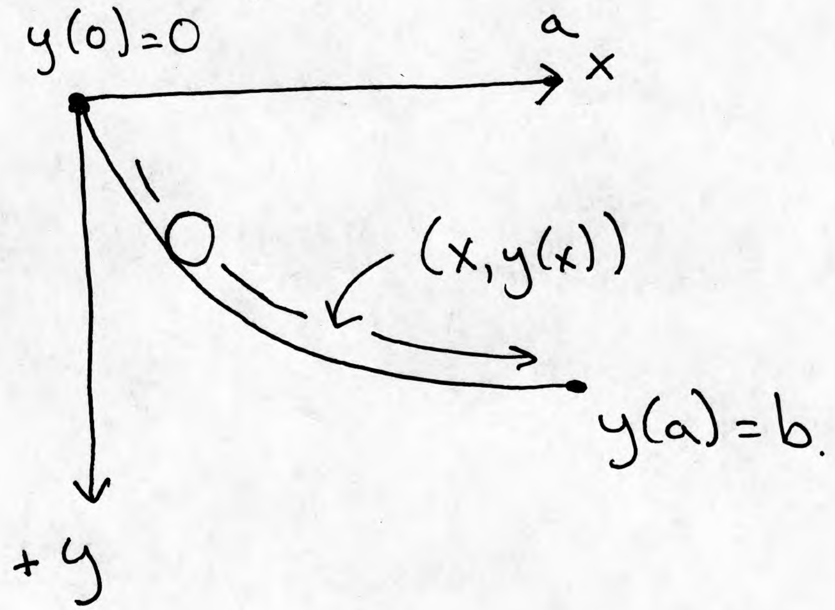
Now this means that $\dfrac{y'(x)}{\sqrt{1+y'(x)^2}}$ is constant and it follows that $y'(x)$ is constant.

Integrating, it follows that

$$\vec{\alpha}(t) = (t, y(t))$$
$$= (t, mt + c)$$

for some constants m and c, and so $\vec{\alpha}(t)$ is a line!

Now we try a classical (more complicated) problem. Suppose we let a weight slide down a curved ramp, starting from rest.



What shape of ramp will let the weight arrive fastest?

We now derive a functional for the travel time. We need some notation.

$Spd(x)$ = speed of the mass when it reaches $(x, y(x))$.

g = acceleration of gravity (a <u>positive</u> number because we flipped the y axis)

$K_E(x)$ = the kinetic energy of the mass when it reaches $(x, y(x))$

We'll take for granted the physics theorem that the ~~change~~ sum of potential and kinetic energy is constant.

Here

$$P_E(x) = \text{potential energy of the mass when it reaches } (x, y(x))$$

$$= -mg\, y(x)$$

while

$$K_E(x) = \frac{1}{2} m \,(Spd(x))^2$$

Since $P_E(0) = K_E(0) = 0$, we have

$$0 = P_E(x) + K_E(x)$$

$$= -mg\, y(x) + \frac{1}{2} m \,(Spd(x))^2$$

We can solve for $Spd(x)$ as a function of $y(x)$.

$$\cancel{m} g\, y(x) = \frac{1}{2} \cancel{m} \left(Spd(x)\right)^2$$

$$2g\, y(x) = Spd(x)^2$$

$$Spd(x) = \sqrt{2g\, y(x)}$$

Again, everything under the square root is <u>positive</u>, and speed is positive.

We can also compute

$$Spd(x) = \left| \frac{d}{dt} \left(x(t), y(x(t)) \right) \right|$$

$$= \left| \left(x'(t), y'(x(t))\, x'(t) \right) \right|$$

$$= x'(t) \left| \left(1, y'(x) \right) \right|$$

$$= x'(t) \sqrt{1 + (y'(x))^2}.$$

Equating these, we can solve for $x'(t)$.

$$x'(t) = \frac{\sqrt{2gy(x)}}{\sqrt{1+(y'(x))^2}}$$

Now the total travel time

$$T = \int_0^a t'(x)\,dx$$

But remember that $t(x)$ is the inverse function of $x(t)$, so $t'(x) = \frac{1}{x'(t)}$. This means

$$T = \int_0^a \frac{1}{x'(t)}\,dx = \int_0^a \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\,dx.$$

We can now find the Euler-Lagrange equation by computing

$$\frac{\partial}{\partial y} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} = \sqrt{1+(y')^2} \cdot \left(-\frac{1}{\cancel{2}}(2gy)^{-3/2} \cdot \cancel{2}g\right)$$

$$= -\frac{g\sqrt{1+(y')^2}}{\left(\sqrt{2gy}\right)^3} = -\frac{1}{2}\frac{1}{\sqrt{2g}}\sqrt{1+(y')^2} \cdot \frac{1}{y^{3/2}}$$

$$\frac{\partial}{\partial y'} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} = \frac{\frac{1}{\cancel{2}}\left(1+(y')^2\right)^{-1/2} \cdot (\cancel{2}y')}{\sqrt{2gy}}$$

$$= \frac{y'}{\sqrt{1+(y')^2}\,\sqrt{2gy}}$$

$$= \frac{1}{\sqrt{2g}}\left(\frac{y'}{\sqrt{1+(y')^2}\,\sqrt{y}}\right).$$

We can now write the E-L equation

$$-\frac{1}{2}\cancel{\frac{1}{\sqrt{2g}}}\sqrt{\frac{1+(y')^2}{y^3}} = \frac{d}{dx}\cdot\cancel{\frac{1}{\sqrt{2g}}}\frac{y'}{\sqrt{y(1+(y')^2)}}$$

Summoning our courage, we compute

$$\frac{d}{dx}\, y'\left(y(1+y'^2)\right)^{-1/2} =$$

$$y''\left(y(1+y'^2)\right)^{-1/2}$$

$$+\, y'\left(-\frac{1}{2}\left(y(1+y'^2)\right)^{-3/2}\right)\left(y'(1+y'^2) + y(2y'y'')\right)$$

$$= \left(y(1+y'^2)\right)^{-1/2}\left[y'' - \frac{1}{2}\frac{y'^2\cancel{(1+y'^2)}}{y\cancel{(1+y'^2)}} - \frac{\cancel{y}(y')^2 y''}{\cancel{y}(1+y'^2)}\right]$$

So the E-L equation becomes

$$-\frac{1}{2}\sqrt{\frac{1+(y')^2}{y^3}}\sqrt{y(1+y'^2)} = y'' - \frac{1}{2}\frac{(y')^2}{y} - \frac{(y')^2 y''}{(1+y'^2)}$$

or

$$-\frac{1}{2}\,\frac{1+y'^2}{y} = y'' - \frac{1}{2}\frac{(y')^2}{y} - \frac{(y')^2 y''}{(1+y'^2)}$$

Now we're going to solve for y''.

$$\cancel{\frac{1}{2}\frac{(y')^2}{y}} - \frac{1}{2}\,\frac{1+\cancel{(y')^2}}{\cancel{y}} = y''\left(1 - \frac{(y')^2}{1+(y')^2}\right)$$

or

$$-\frac{1}{2y} = y''\left(\frac{1+\cancel{(y')^2}-\cancel{(y')^2}}{1+(y')^2}\right)$$

so

$$y'' = -\frac{1+y'^2}{2y}.$$

And we've reached another milestone: this is a differential equation of 2nd order for y.

Now we need to solve the differential equation! It helps to multiply through by $2yy'$ to get

$$2yy'y'' = -y' - (y')^3$$

or

$$y' + 2yy'y'' + (y')^3 = 0.$$

or (this is the clever bit!)

$$\frac{d}{dx}\left(y + y(y')^2\right) = 0.$$

It follows that

$$y + y(y')^2 = C$$

for some constant C. We can solve for y' to get a new differential equation

$$y' = \sqrt{\frac{C-y}{y}}$$

In homework, you'll see that this can be solved by our standard method to get

$$x(t) = C\left(t - \frac{1}{2}\sin 2t\right)$$
$$y(t) = C\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)$$

which we recognize as a cycloid!