Arclength, and rectifiable curves.

We are used to defining length by integrating speed:

Definition. If $\gamma : [a,b] \to \mathbb{R}^3$ is a C^1 curve, the length of the portion of γ between a and t is

$$s(t) = \int_a^t |\gamma'(t)|\, dt.$$

What if γ is continuous, but not C^1 (maybe it has corners)? Does it still _have_ a length function? What does length mean?

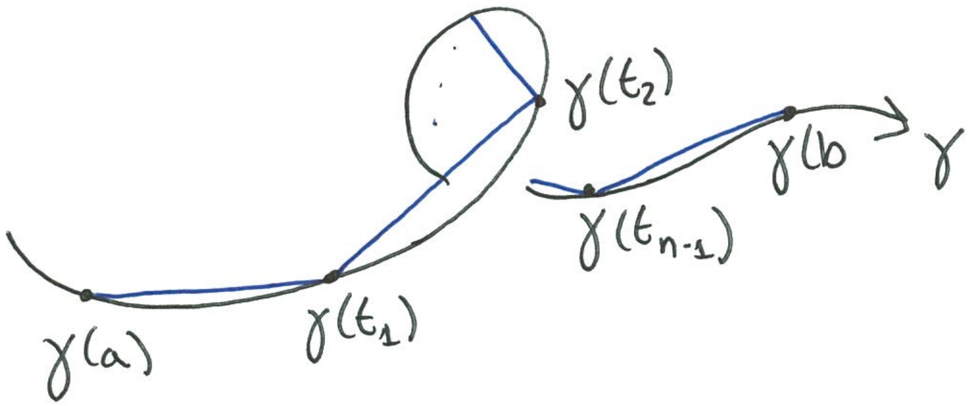
We can make a natural definition:

Definition. Given a partition

$$P = \{a = t_0 < t_1 < \dots < t_n = b\}$$

of $[a,b]$, let

$$\ell(\gamma, P) = \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})|$$



It's clear that $\ell(\gamma, P)$ is the length of the polygon formed by connecting the $\gamma(t_i)$.

We let

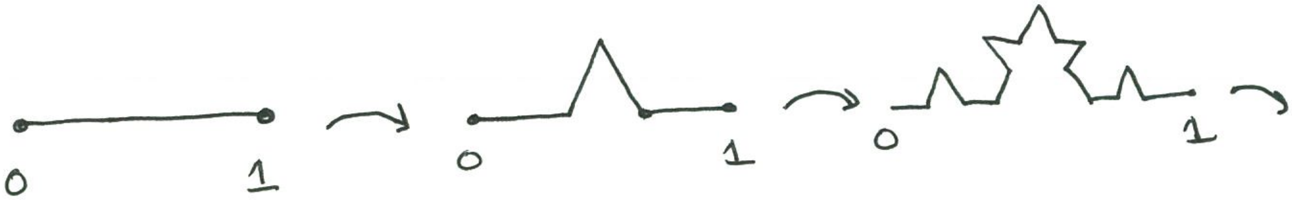
$$\text{length}(\gamma) = \sup_P \ell(\gamma, P)$$

That is, the least upper bound on the (infinite) set of polygons inscribed in γ.

lengths of

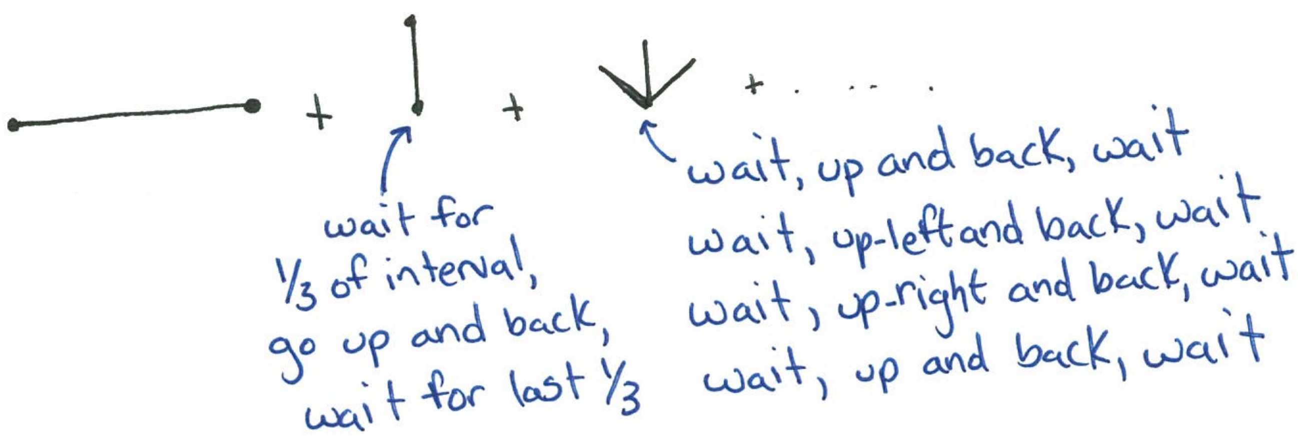
Definition. If length(γ) exists, then we say γ is <u>rectifiable</u> (technically, as a function, γ is a <u>function of bounded variation</u>).

Example. The Koch snowflake is the limit curve obtained by replacing the middle third of each segment of a polygon with 2 sides of an equilateral triangle.



Proposition. The Koch snowflake is continuous, but not rectifiable.

Proof. Notice that the curve is the sum of an infinite series of vector functions.



Since each subsection is scaled down by a factor of 3, these functions have norms bounded by

$$\alpha, \frac{\alpha}{3}, \frac{\alpha}{9}, \ldots, \frac{\alpha}{3^n}, \ldots$$

This is a convergent series of bounds, so the limit curve ~~converges~~ is continuous by the Weierstrauss M-test.

The intermediate polygons are all inscribed in the final curve, and their lengths scale up by $4/3$ at each step; since $4/3, (4/3)^2, \ldots, (4/3)^n$ is unbounded, γ is not rectifiable. $\square$

In the other direction,

Proposition. Every C^1 curve γ is rectifiable and has $\text{length}(\gamma) = \int_a^b |\gamma'(t)|\,dt$.

Proof. It's a homework exercise

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

for any vector-valued function $f(x)$.

Using this, we can show that for any partition P, we have

$$\begin{aligned} \ell(\gamma, P) &= \sum |\gamma(t_i) - \gamma(t_{i-1})| \\ &= \sum \left| \int_{t_{i-1}}^{t_i} \gamma'(t)\,dt \right| \\ &\leq \sum \int_{t_{i-1}}^{t_i} |\gamma'(t)|\,dt \\ &= \int_a^b |\gamma'(t)|\,dt. \end{aligned}$$

(That is, any inscribed polygon is no longer than the curve.) Now if we take

$$s(t) = \sup_P \{\ell(\gamma, P) \mid P \text{ partitions } [a,t]\}$$

then we have just shown

$$s(t+h) - s(t) \leq \int_t^{t+h} |\gamma'(t)|\,dt.$$

Now

$$s(t+h) - s(t) = \sup_P \left\{ \ell(\gamma, P) \,\middle|\, P \text{ partitions } [t, t+h] \right\}$$

Since $\{t, t+h\}$ is certainly one such partition, and has length $|\gamma(t+h) - \gamma(t)|$, we have

$$|\gamma(t+h) - \gamma(t)| \leq s(t+h) - s(t) \leq \int_t^{t+h} |\gamma'(t)|\,dt$$

Dividing through by h and taking the limit as $h \to 0$, we get

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = |\gamma'(t)|.$$

We conclude that s is differentiable and that $s'(t) = |\gamma'(t)|$. Integrating,

$$s(b) = \text{length } \gamma = \int_a^b |\gamma'(t)|\,dt$$

as desired. $\square$

If $|\gamma'(t)| = 1$ for all t, then

$$s(t) = t - a$$

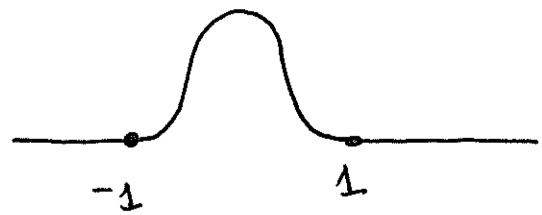
and we say γ is <u>parametrized</u> <u>by</u> <u>arclength</u>. (We usually write $\gamma(s)$ for such curves.)

Example. $\gamma(t) = (\cos t, \sin t)$ is parametrized by arclength.

~~Theorem.~~

We now need a new definition. When does a parametrized curve have corners? It's tempting to think "where γ is not differentiable", but that's not the whole story.

Consider

$f(t) =$ 

$$= \begin{cases} \cos(\frac{\pi}{\cancel{2}} t) + 1, & t \text{ in } [-1,1] \\ 0, & \text{otherwise} \end{cases}$$

This is centainly differentiable. But

$$\gamma(t) = (f(t), f(t-2))$$

~~has the~~ traces out  (!)

The problem is that $\gamma'(1) = \vec{0}$ at the corner.

Definition. γ is <u>regular</u> if $|\gamma'(t)| \neq 0$.

Then we have

Proposition. A regular curve $\gamma(t)$ has no corners.

and, further,

Theorem. If $\gamma(t)$ is a regular curve, there is some differentiable function $t(s)$ so that $\gamma(t(s)) = \gamma(s)$ is parametrized by arclength.

Proof. Since $s(t) = \int_a^t |\gamma'(t)|\, dt$ is differentiable and strictly increasing, it has an inverse function $t(s)$.

$$\frac{d}{ds}\gamma(t(s)) = \gamma'(t(s)) \cdot \frac{dt}{ds}$$

$$= \frac{\gamma'(t(s))}{ds/dt}$$

derivative of inverse function!

$$= \frac{\gamma'(t(s))}{|\gamma'(t(s))|}$$

This has $\left| \dfrac{\gamma'(s)}{|\gamma'(s)|} \right| = 1$, as desired. $\square$

Now this should be enough to make you wonder if a <u>rectifiable</u> curve can be arclength parametrized. The~~re~~ answer is "yes":

Theorem. A rectifiable curve is almost everywhere differentiable, and has derivative $\gamma'(t)$ which exists as a Radon measure, and can be written as $\gamma(s)$.

But rigorously understanding these terms will lead to a couse in Real Analysis...