Arclength, regular curves.

Now that we've seen a few examples of parametrized curves, our goal is to define a class of curves to study.

Definition.

A parametrized curve $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$ is:

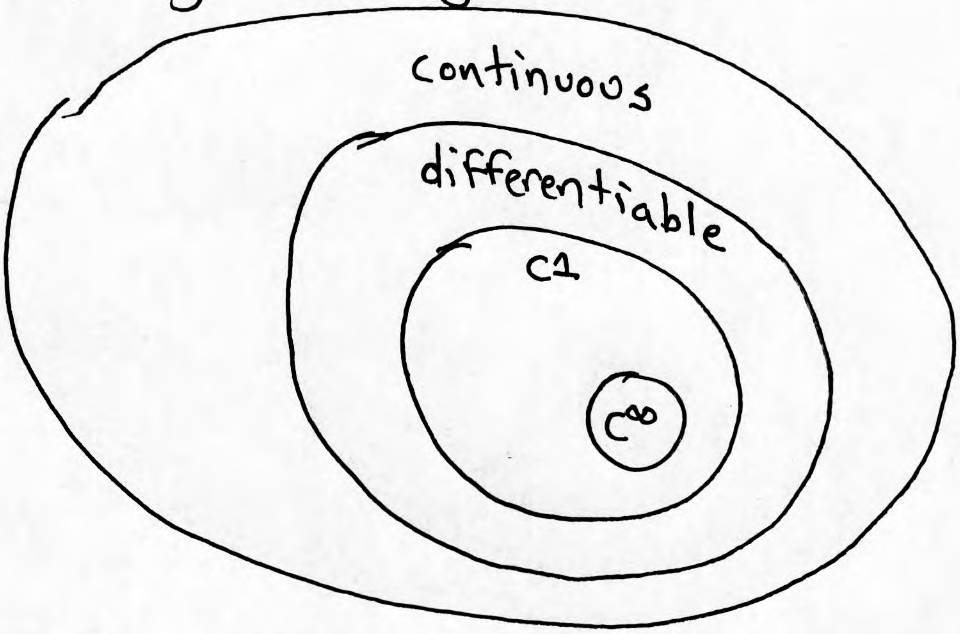
1) continuous (or C^0) if $\vec{\alpha}$ is a continuous function.

2) differentiable if $\vec{\alpha}'(t)$ exists for all t

3) continuously differentiable (or C^1) if $\vec{\alpha}'(t)$ is a continuous function.

4) smooth (or C^∞) if the derivative $\frac{d^n}{dt^n}\vec{\alpha}(t)$ is a continuous function of t for all n.

As you might expect, these conditions are progressively stricter:



We will mostly study C^1 curves in this course.

Calculus fact. The graph of a C^1 function $y(x)$ has no corners.

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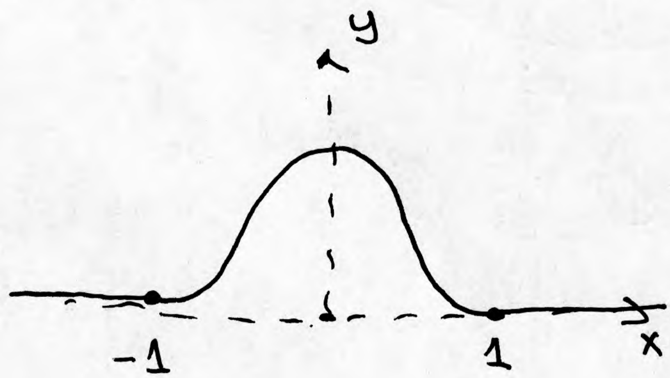
1 It's hard to imagine a curve that's differentiable everywhere, but has a discontinuous derivative. $f(x) = \{ x^2\sin(1/x), x \neq 0, 0, x=0 \}$ has this property at 0.

Here is a surprising example.

Consider the function

$$f(t) = \begin{cases} \cos(\pi t) + 1, & t \in [-1,1] \\ 0, & \text{otherwise.} \end{cases}$$

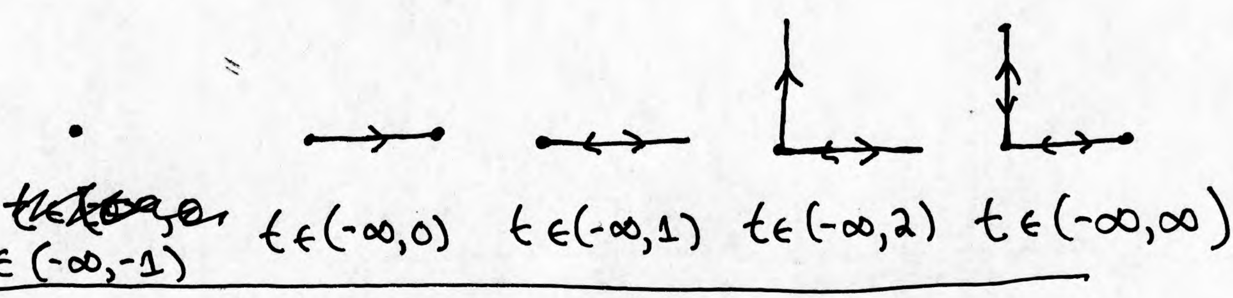
This is a C^1 function[1], with graph



The parametrized curve

$$\vec{\alpha}(t) = (f(t), f(t-2))$$

traces out



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1. Just check that $f'(-1) = f'(1) = 0$.

which has a corner! At the corner,

$$\vec{\alpha}'(t) = (f'(t), f'(t-2))$$

is given by $\vec{\alpha}'(1) = (f'(1), f'(-1)) = (0,0)$.

This leads us to the definition

Definition. A C^1 curve $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$ is <u>regular</u> if $\|\vec{\alpha}'(t)\| > 0$ for all t.

A regular parametrized curve has no corners.[1]

Definition. A C^1 curve $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$ is <u>unit speed</u> if $\|\vec{\alpha}'(t)\| = 1$ for all t.

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1. I'm not stating this as a theorem because we don't have a rigorous definition of "corner."

Theorem. If $\vec{\alpha}(t)$ is a regular curve, there exists some differentiable function $p(s)$ so that $\vec{\alpha}(p(s))$ is unit speed.

Proof. We start by defining

$$q(s) = \int_0^s \|\vec{\alpha}'(t)\| \, dt$$

By the fundamental theorem of calculus,

$$q'(s) = \|\vec{\alpha}'(s)\| > 0,$$

so $q'(s)$ is a continuous, positive function, and $q(s)$ is a C^1 function which is strictly increasing.

By the inverse function theorem, $q(s)$ has an inverse function $p(s)$.

Recall from homework that

$$q(p(s)) = s$$

so (differentiating w.r.t. s)

$$q'(p(s))\, p'(s) = 1$$

or

$$p'(s) = \frac{1}{q'(p(s))} = \frac{1}{\|\vec{\alpha}'(p(s))\|}.$$

Thus

$$\frac{d}{ds}\vec{\alpha}(p(s)) = \vec{\alpha}'(p(s)) \cdot p'(s)$$

$$= \frac{\vec{\alpha}'(p(s))}{\|\vec{\alpha}'(p(s))\|}$$

has unit norm and so $\vec{\alpha}(p(s)) = \vec{\alpha}(s)$ is unit speed.

Definition. We call $\vec{\alpha}(s)$ the "<u>unit speed reparametrization</u>" of $\vec{\alpha}(t)$.

In single-variable calculus, we learn to recite:

"If $x(t)$ is position, then $x'(t)$ is velocity and $|x'(t)|$ is speed. The integral of velocity is displacement, and the integral of speed is distance traveled."

"Theorem." The length of the portion of $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$ between $\vec{\alpha}(a)$ and $\vec{\alpha}(b)$ is given by

$$\text{length}(\vec{\alpha}; a, b) = \int_a^b \|\vec{\alpha}'(t)\|\, dt,$$

<u>if $\vec{\alpha}(t)$ is a C^1 curve.</u>

We can't prove this theorem (yet) but if we accept it, we see:

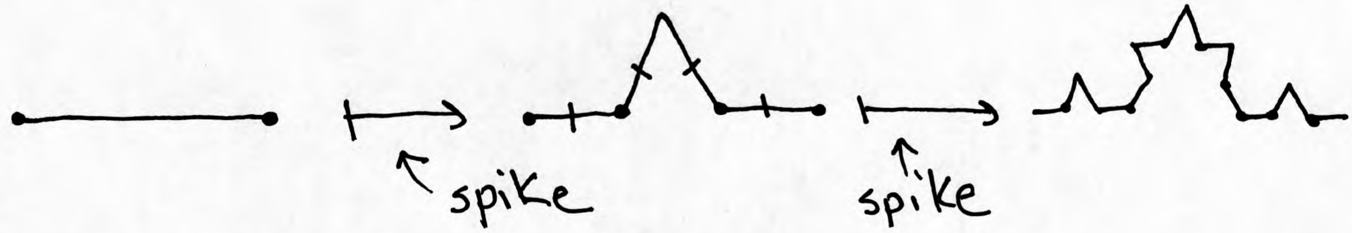
Corollary. If $\vec{\alpha}(s)$ is unit speed, then

$$\text{length}(\vec{\alpha}; a, b) = b - a.$$

For this reason, we often call $\vec{\alpha}(s)$ an "arclength parametrization" of $\vec{\alpha}$.

Now we are going to go a little deeper. The reason that we can't prove the last theorem is disconcerting: we have no idea what length <u>is</u>! (Especially for C^0 curves...)

Example. Here is an example of a (continuous) curve[2], defined as a limit.



We define the operation "spike" to replace every line segment with a 4 segment ~~coarse~~ polygon given by cutting out the middle third and replacing it by two segments of the same length.

$\vec{\alpha}: \mathbb{R} \to \mathbb{R}^2$ is the limit

$$\lim_{n\to\infty} \text{spike}^{[n]\,1}\big(\underset{0}{\bullet}\!\!-\!\!-\!\!-\!\!\underset{1}{\bullet}\big)$$

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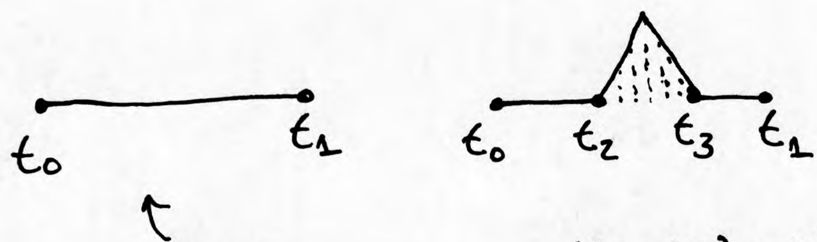
1 spike(spike(. . . . . spike($\bullet\!-\!\bullet$)) . . .) = spiake$^{[n]}$($\bullet\!-\!\bullet$)

2. This called the Koch snowflake curve. It's a fractal!

If $\vec{\beta}(t)$ is parametrized, we can parametrize spike$(\vec{\beta}(t))$ by



$\vec{\beta}(t)$ spike$(\vec{\beta}(t))$

where spike$(\vec{\beta}(t)) = \vec{\beta}(t)$ for $t \in [t_0, t_2], [t_3, t_1]$ but spike$(\vec{\beta}(t))$ is the corresponding point on the spike for $t \in [t_2, t_3]$.

Fact[1]. The limit of these parametrizations of spike$^{[n]}(\underset{0}{\bullet}\!-\!\!-\!\underset{1}{\bullet})$ is a continuous parametrization of $\vec{\alpha}(t)$.

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1. This can be proved by the Weierstrauss M-test, but that's a problem for graduate students.

It is clear that $\vec{\alpha}(t)$ is not differentiable at infinitely many points.

Does $\vec{\alpha}(t)$ have a length?

Definition. If $\ell(\vec{p},\vec{q})$ is the line segment from $\vec{p}$ to $\vec{q}$ in $\mathbb{R}^n$, we define $\text{length}(\ell) := \|\vec{q}-\vec{p}\|$.

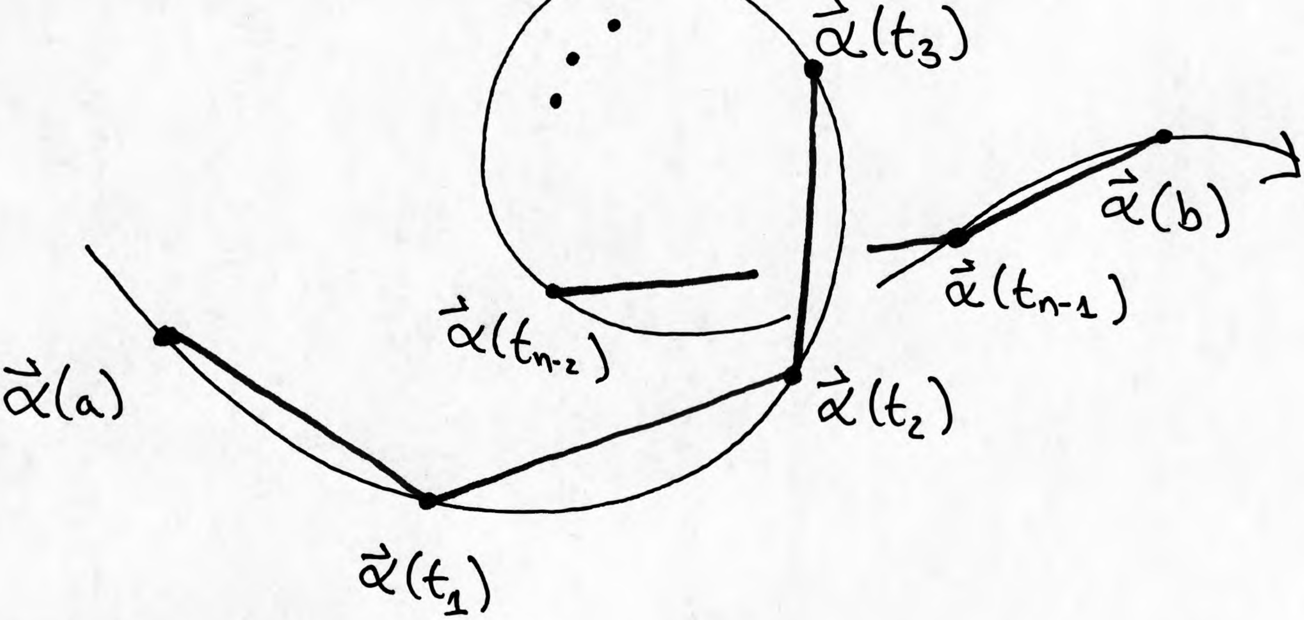
Definition[1] If S is any set of real numbers, $\sup S = \text{lub } S$ is the smallest M so that $M \geq s$ for every $s \in S$.

Theorem[2]: If S is a set of real numbers with a finite upper bound M, then $\sup S$ exists and is unique.

1,2) Both of these come from Real Analysis - a class that I encourage you to take!

Definition. If $\vec{\alpha} : \mathbb{R} \to \mathbb{R}^n$ is any[1] parametrized curve, and $P = \{ a = t_0 < t_1 < \dots < t_n = b \}$, we say that the <u>length of the inscribed polygon given by</u> P is

$$\text{length}(\vec{\alpha}; P) = \sum_{i=1}^{n} \| \vec{\alpha}(t_i) - \vec{\alpha}(t_{i-1}) \|$$



We see that $\text{length}(\vec{\alpha}; P)$ is the sum of the lengths of the edges of the polygon joining $\vec{\alpha}(t_0), \dots, \vec{\alpha}(t_n)$.

Now we can (finally!) define length.

Definition. If $\sup_{p} \text{length}(\vec{\alpha}; p) = L < \infty$

then we say $\vec{\alpha}$ is a <u>rectifiable curve</u>[1] on $[a,b]$ and that L is the <u>length</u> of the portion of $\vec{\alpha}$ between $\vec{\alpha}(a)$ and $\vec{\alpha}(b)$.

Proposition. The snowflake curve is continuous, but not rectifiable.

Notice that spike doesn't change the endpoints of any segment.

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1. Rectifiable sets are beautiful objects with a rich theory – see Morgan or Federer, <u>Geometric Measure Theory</u>.

Thus all of the intermediate polygons

$\text{spike}^{[n]}(\bullet\!\!-\!\!\bullet)$ are inscribed in the

limit curve $\vec{\alpha}(t)$.

But the length of $\text{spike}(\bullet\!\!-\!\!\bullet) =$ 

is $4/3$ of the length of the original

line segment $\bullet\!\!-\!\!\bullet$, so the length of

$\text{spike}^{[n]}(\bullet\!\!-\!\!\bullet) = (4/3)^n \text{ length}(\bullet\!\!-\!\!\bullet)$. Since

$(4/3)^n \to \infty$, there is no finite upper

bound for $\text{length}(\vec{\alpha}; \mathcal{P})$ and so

$\vec{\alpha}$ is <u>not</u> rectifiable. $\square$

We can now prove

Theorem. If $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$ is a C^1 curve then $\vec{\alpha}$ is a rectifiable curve and

$$\operatorname{length}(\vec{\alpha}; a, b) = \int_a^b \|\vec{\alpha}'(t)\| \, dt.$$

Proof. Recall from our first set of notes that for any $\vec{\beta}: \mathbb{R} \to \mathbb{R}^n$ we had proved

$$\left\| \int_a^b \vec{\beta}(t) \, dt \right\| \leq \int_a^b \|\vec{\beta}(t)\| \, dt.$$

Thus we can write

$$\operatorname{length}(\vec{\alpha}; p) = \sum_{i=1}^{n} \|\vec{\alpha}(t_i) - \vec{\alpha}(t_{i-1})\|$$

$$= \sum_{i=1}^{n} \left\| \int_{t_{i-1}}^{t_i} \vec{\alpha}'(t) \, dt \right\| \qquad \text{(because } \vec{\alpha} \text{ is } C^1\text{)!}$$

$$\leq \sum_{i=1}^{\infty} \int_{t_{i-1}}^{t} \|\vec{\alpha}'(t)\| dt = \int_a^b \|\vec{\alpha}'(t)\| dt.$$

Since $\mathcal{P}$ was arbitrary, this proves that $\int_a^b \|\vec{\alpha}'(t)\| dt$ (which is <u>finite</u> b/c $\|\vec{\alpha}'(t)\|$ is continuous) is an upper bound for $\{\text{length}(\vec{\alpha}; \mathcal{P})\}$.

Thus (by our theorem)

$$\sup_{\mathcal{P}} \text{length}(\vec{\alpha}; \mathcal{P}) = \text{length}(\vec{\alpha}; a, b)$$

exists and is $\leq \int_a^b \|\vec{\alpha}'(t)\| dt$ for all $b > a$. Further, $\vec{\alpha}(t)$ is a rectifiable curve.

Now let's define a new function

$$s(t) = \text{length}(\vec{\alpha}; a, t)$$

You'll prove in homework that

$$s(t+h) - s(t) = \text{length}(\vec{\alpha}; t, t+h)$$

So we have

$$s(t+h) - s(t) \leq \int_t^{t+h} \|\vec{\alpha}'(t)\| \, dt$$

Further, since

$$\text{length}(\vec{\alpha}; t, t+h) = \sup_{P} \text{length}(\vec{\alpha}; P)$$

for all partitions $P = \{t = t_0 < t_1 < \dots < t_n = t+h\}$
we also have

$$\|\vec{\alpha}(t+h) - \vec{\alpha}(t)\| \leq s(t+h) - s(t)$$

Thus

$$\left\| \frac{\vec{\alpha}(t+h) - \vec{\alpha}(t)}{h} \right\| \le \frac{s(t+h) - s(t)}{h} \le \frac{1}{h} \int_t^{t+h} \|\vec{\alpha}'(t)\| \, dt$$

Taking the limit as $h \to 0^+$ proves[1] that

$$s'(t) = \|\vec{\alpha}'(t)\|$$

& Thus (in particular) $s(t)$ is a C^1 function and

$$\begin{aligned} \text{length}(\vec{\alpha}; a, b) &= s(b) - s(a) \\ &= \int_a^b s'(t) \, dt \\ &= \int_a^b \|\vec{\alpha}'(t)\| \, dt \end{aligned}$$

which completes the proof! □

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1. OK, if $h \to 0^-$, we have to make minor adjustments. But that also works.