Adjacency matrices, interlacing and the Perron-Frobenius theorem

We recall

Definition. The adjacency matrix M_G of a graph is given by

$$\left(M_G \vec{x}\right)(\vec{v}_1) = \sum_{\{v_1, v_2\}} \vec{x}(v_2)$$

The notation deserves a moment of explanation. M_G is a $\mathcal{V} \times \mathcal{V}$ matrix, so it operates on vectors $\vec{x} \in \mathbb{R}^{\mathcal{V}}$, returning vectors $(M_G \vec{x}) \in \mathbb{R}^{\mathcal{V}}$.

If v_1 is some vertex in G, and $\vec{y}$ is any vector in $\mathbb{R}^{\mathcal{V}}$, then

"$\vec{y}(v_1)$" is the component of $\vec{y}$

in whatever position corresponds to vertex v_1.

The sum $\sum_{v_1 - v_2}$ is the sum over all v_2 which are connected to v_1 by an edge.

Definition. The eigenvalues of M_G are denoted $\mu_1 \geq \dots \geq \mu_n$.

This is <u>different</u> from our convention that the eigenvalues of <u>L_G</u> are denoted $\lambda_1 \leq \dots \leq \lambda_n$.

Let's see why this is a good idea.

Definition. A graph G is <u>d-regular</u> if every vertex has degree d.

Proposition. If G is a d-regular graph then $\lambda_i = d - \mu_i$.

Proof. We know $L_G = D_G - M_G$. If G is d-regular then $D_G = dI$. So the i-th eigenvector $\vec{\psi}_i$ of L_G has

$$M_G \vec{\psi}_i = (D_G - L_G)\vec{\psi}_i = (dI - L_G)\vec{\psi}_i = (d - \lambda_i)\vec{\psi}_i$$

and is an eigenvector of M_G with eigenvalue $d-\lambda_i$. Since this is true for all $\vec{\psi}_1, \ldots, \vec{\psi}_v$, these must be all of the v eigenvalues of M_G. Further, since $\lambda_1 \leq \ldots \leq \lambda_n$, we have $d-\lambda_1 \geq \ldots \geq d-\lambda_n$ and so $\mu_i = d - \lambda_i$ as claimed. $\square$

There's a suggestive corollary of this result:

Cor. If G is d-regular, then $\mu_1 = d$ and the corresponding eigenvector is constant.

Proof. We Know $\lambda_1 = 0$, with $\vec{\varphi}_1$ constant...

So what if G is <u>not</u> d-regular? Well,

Lemma. If d_{ave} is the average degree of a vertex in G and d_{max} is the maximum degree of a vertex in G,

$$d_{ave} \leq \mu_1 \leq d_{max}.$$

Proof. We know from Courant-Fischer[1]

$$\mu_1 = \max_{\vec{x}} \frac{\langle \vec{x}, M_G \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle}$$

$$\geq \max_{\vec{1}} \frac{\langle \vec{1}, M_G \vec{1} \rangle}{\langle \vec{1}, \vec{1} \rangle}$$

$$= \frac{\left\langle \vec{1}, \begin{bmatrix} d_1 \\ \vdots \\ d_v \end{bmatrix} \right\rangle}{v} = \frac{\sum_i d_v}{v} = d_{ave}.$$

The interesting step here is

$$(M_G \vec{1})(v_1) = \sum_{v_1 \sim v_2} \vec{1}(v_2) = \sum_{v_1 \sim v_2} 1 = \underset{\substack{\uparrow \\ \text{degree}}}{d(v_1)}.$$

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1. This is a standard trick for extracting infomation about eigenvalues - use the Rayleigh quotient with a cleverly chosen vector. We'll see this a lot.

Now suppose $\vec{\varphi}_1$ is the eigenvector of M_G with eigenvalue μ_1. W.l.o.g. we can assume that some entry of $\vec{\varphi}_1$ is positive. Suppose v_{max} has $\vec{\varphi}_1(v_{max}) \geq \vec{\varphi}_1(v)$ for all v. Now

$$\mu_1 = \frac{(M_G \vec{\varphi}_1)(v_{max})}{\vec{\varphi}_1(v_{max})} = \frac{\sum_{v_{max} \sim v_2} \vec{\varphi}_1(v_2)}{\vec{\varphi}_1(v_{max})}$$

$$= \sum_{v_{max} \sim v_2} \frac{\vec{\varphi}_1(v_2)}{\vec{\varphi}_1(v_{max})}$$

$$\leq \sum_{v_{max} \sim v_2} 1 = d(\vec{v}_{max}) \leq d_{max}. \qquad \square$$

We note that this might be a little stronger, than claimed if $d(\vec{v}_{max}) \neq d_{max}$.

If G is d-regular, then $d_{ave} = d_{max} = d$, so this implies our previous lemma.

In fact, we can get a converse result!

Lemma. If G is connected and $\mu_1 = d_{max}$, then G is d_{max}-regular.

Proof. If $\mu_1 = d_{max}$ then (by our previous argument) we have

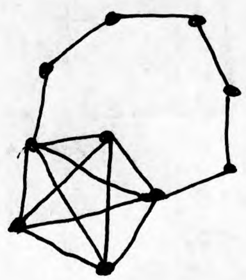
$$\sum_{v_{max} \leftrightarrow v_2} \frac{\vec{\varphi}_1(v_2)}{\varphi_1(v_{max})} = \sum_{v_{max} \leftrightarrow v_2} 1 = d(\vec{v}_{max}) = d_{max}.$$

Therefore $\vec{\varphi}_1(v_2) = \vec{\varphi}_1(v_{max})$ for all v_2 joined to v_{max} by an edge and $d(\vec{v}_{max}) = d_{max}$.

But then all these v_2 are also maxes for φ_1 and we can repeat the argument to show their degrees are also d_{max}.

Since G is connected, this eventually covers the entire graph. □

Can we do even better? For instance, consider

 $= G$

which has a high average degree some places, but a low average degree in others.

Cauchy's Interlacing Theorem.

Let A be an $n \times n$ symmetric matrix and B be the $(n-1)\times(n-1)$ submatrix obtained by deleting the same row and column[1]. Let $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_{n-1}$ be the eigenvalues of A and B. Then

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \beta_{n-1} \geq \alpha_n .$$

Proof. There's a proof in the book, which is basically to apply Courant-Fischer. □

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1. This is called a <u>principal</u> submatrix, as you might remember.

Definition. If S is a subset of the vertices of a graph G then $G(S)$, called "the subgraph induced by S" is the graph with vertex set S and edge set $\{ v_1 \bullet\!\!-\!\!\bullet v_2 \mid v_1, v_2 \in S \text{ and } v_1 \bullet\!\!-\!\!\bullet v_2 \text{ an edge of } G \}$.

Definition. If M is an $n \times n$ symmetric matrix and $S \subset \{1, \ldots, n\}$ then $M(S)$ is the symmetric submatrix with rows and columns in S.

Lemma. For every $S \subset \overset{\text{vertices of } G}{V(G)}$, let $d_{ave}(S)$ be the average vertex degree of $G(S)$. Then

$$d_{ave}(S) \leq \mu_1$$

(where μ_1 is $\mu_1(G)$).

Proof. If M_G is the adjacency matrix of G, then $M(S)$ is the adjacency matrix of $G(S)$.

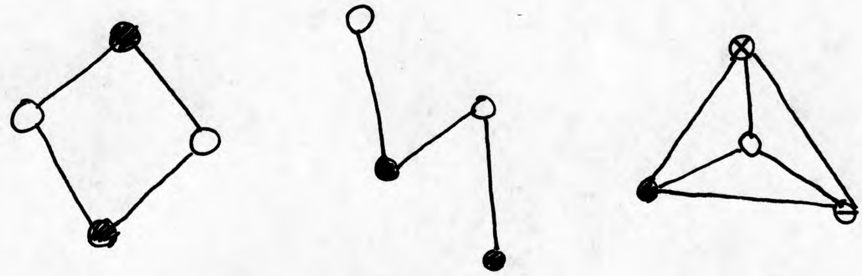
Now if $(\mu_S)_1$ is the largest eigenvalue of $G(S)$, by Cauchy interlacing,

$$(\mu_S)_1 \leq \mu_1$$

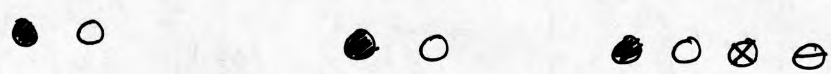
But we already know that

$$d_{ave}(G(S)) \leq (\mu_S)_1 \leq \mu_1 . \qquad \square$$

Definition. A <u>coloring</u> of a graph is an assignment of colors to vertices so that every pair of vertices joined by an edge have different colors.



2 colors 2 colors 4 colors



Definition. A graph is <u>K-colorable</u> if it can be colored with only K colors. The <u>chromatic number</u> $\chi(G)$ of a graph is the smallest K so that G is K-colorable.

Lemma. $\chi(G) \leq d_{max} + 1$.

Proof. Assign colors to vertices one by one, and proceed by induction, with the hypothesis "no two adjacent vertices have the same color".

Base case. 0 ~~colors~~ vertices colored. True.

Inductive step. Choose an uncolored vertex. It has at most d_{max} neighbors with at most d_{max} different colors. ~~Use~~ At least one color is left unused. Color the ~~new~~ vertex with an unused color. □

Wilf's Theorem. $\chi(G) \leq \lfloor \mu_1 \rfloor + 1$.

Proof. We will induct on the number of vertices in the graph.

Base case. The one vertex graph with no edges has $\chi(G) = 1$ and $\mu_1 = \lambda_1 + 0 = 0$, so $\chi(G) \leq \lfloor \mu_1 \rfloor + 1 = 1$.

Inductive step. Suppose the theorem is true for all graphs with $v-1$ vertices and let G be a graph with v vertices.

Since $d_{ave} \leq \mu_1 \leq d_{max}$, we have $d_{ave} \leq \lfloor \mu_1 \rfloor$ and there is a vertex v_* of G with degree $\leq \lfloor \mu_1 \rfloor$.

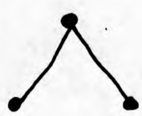
vertices of G

Let $S = V(G) - v$. By Cauchy interlacing,

$$\mu_1(G(S)) \leq \mu_1(G)$$

so $\lfloor \mu_1(G(S)) \rfloor \leq \lfloor \mu_1(G) \rfloor$. Further, by induction, $G(S)$ has a coloring with at most $\lfloor \mu_1(G(S)) \rfloor + 1 \leq \lfloor \mu_1 \rfloor + 1$ colors.

Now v has degree at most $\lfloor \mu_1 \rfloor$ so no more than $\lfloor \mu_1 \rfloor$ of the colors are used by neighbors of v, and at least one is left to color v. $\square$

Example.  has $\mu_1 = \sqrt{2}$, but $d_{max} = 2$. Thus Wilf's theorem tells us that $\chi(G) \leq \lfloor \sqrt{2} \rfloor + 1 = 2$, while $d_{max} + 1 = 3$.

Perron-Frobenius Theorem.

Let G be a connected (weighted) graph, let M_G be its adjacency matrix, and let $\mu_1 \geq \cdots \geq \mu_v$ be the eigenvalues of M_G. Then

a. μ_1 has a strictly positive eigenvector

b. $\mu_1 \geq -\mu_n$

c. $\mu_1 > \mu_2$

In order to prove this, we'll need

Lemma. Let G be a connected graph, let M_G be its adjacency matrix and let some non-negative nonzero $\vec{\varphi}$ be an eigenvector of M_G. Then $\vec{\varphi}$ is strictly positive.

Proof. If the set $\{a \in V(G) \mid \vec{\varphi}(a) = 0\}$ is nonempty, there is some edge $b \bullet\!\!-\!\!\bullet c$ so that $\vec{\varphi}(b) = 0$ and $\vec{\varphi}(c) > 0$.

After all, we know $\{c \in V(G) \mid \vec{\varphi}(c) > 0\}$ is nonempty (b/c $\vec{\varphi}$ is nonzero) and G is connected.

Now

$$\nu \vec{\varphi}(b) = (M\vec{\varphi})(b) = \sum_{b \bullet\!-\!\bullet v} \vec{\varphi}(\vec{v}) \geq \vec{\varphi}(c) > 0$$

The contradiction proves $\{a \in V(G) \mid \vec{\varphi}(a) = 0\}$ must have been empty. $\square$

Proof. (of Perron-Frobenius)

Let $\vec{\varphi}_1$ be an eigenvector of M_G with eigenvalue μ_1, and assume $\|\vec{\varphi}_1\| = 1$. Let $\vec{x} = |\varphi_1|$.

We claim $\vec{x}$ is an eigenvector of M_G with eigenvalue μ_1.

Now $\langle \vec{x}, \vec{x} \rangle = \langle \vec{\varphi}_1, \vec{\varphi}_1 \rangle \overset{=1}{}$. Further,

$$\mu_1 = \langle \vec{\varphi}_1, M_G \vec{\varphi}_1 \rangle = \sum_{\substack{a,b \\ \text{in } V(G)}} M_G(a,b)\, \vec{\varphi}_1(a)\, \vec{\varphi}_1(b)$$

$$\leq \sum_{\substack{a,b \text{ in} \\ V(G)}} M_G(a,b)\, |\vec{\varphi}_1(a)|\, |\vec{\varphi}_1(b)|$$

$$= \langle \vec{x}, M_G \vec{x} \rangle$$

Since μ_1 is the maximum possible

Rayleigh quotient $\langle \vec{v}, M_G \vec{v} \rangle$ for <u>any</u> unit vector $\vec{v}$, the Courant-Fischer theorem says that $\langle \vec{x}, M_G \vec{x} \rangle = \mu_1$ and that $\vec{x}$ is an eigenvector of M_G with eigenvalue μ_1.

Since $\vec{x}$ is non-negative, our lemma tells us $\vec{x}$ is positive, proving a.

We now claim $\mu_1 \geq -\mu_n$.

Let $\vec{\varphi}_n$ be the eigenvector of M_G with eigenvalue μ_n, and let $\vec{y} = |\vec{\varphi}_n|$, and $\|\vec{y}\| = \|\vec{\varphi}_n\| = 1$. Now

$$|\mu_n| = |\langle \vec{\varphi}_n, M_G \vec{\varphi}_n \rangle|$$

$$= \Big| \sum_{\substack{a,b \\ \in V(G)}} M_G(a,b)\, \vec{\varphi}_n(a)\, \vec{\varphi}_n(b) \Big|$$

$$\leq \sum_{\substack{a,b \\ \in V(G)}} \underbrace{M_G(a,b)}_{\hookrightarrow \text{ already } \geq 0 \text{ by defn.}} |\vec{\varphi}_n(a)||\vec{\varphi}_n(b)|$$

$$= \sum_{\substack{a,b \\ \in V(G)}} M_G(a,b)\, \vec{y}(a)\, \vec{y}(b)$$

$$= \underbrace{\langle \vec{y}, M\vec{y} \rangle}_{\substack{\text{Rayleigh quotient of } \vec{y} \\ \text{since } \|\vec{y}\| = 1}} \leq \mu_1.$$

which completes the proof of b.

We now claim $\mu_2 < \mu_1$. Suppose that $\vec{\varphi}_2$ is an eigenvector of M_G of eigenvalue μ_2, which is orthogonal to $\vec{\varphi}_1$ and has $\|\vec{\varphi}_2\| = 1$. As before, let $\vec{y} = |\vec{\varphi}_2|$, and note $\|\vec{y}\| = \|\vec{\varphi}_2\|$.

We have

$$\mu_2 = \langle \vec{\varphi}_2, M_G \vec{\varphi}_2 \rangle = \sum_{\substack{a,b \\ \in V(G)}} M_G(a,b)\, \vec{\varphi}_2(a)\, \vec{\varphi}_2(b)$$

$$\leq \sum_{\substack{a,b \\ \in V(G)}} M_G(a,b)\, |\vec{\varphi}_2(a)|\, |\vec{\varphi}_2(b)|$$

$$= \sum_{\substack{a,b \\ \in V(G)}} M_G(a,b)\, \vec{y}(a)\, \vec{y}(b)$$

$$= \langle \vec{y}, M_G \vec{y} \rangle \leq \mu_1 .$$

Now if $\mu_2 = \mu_1$, then (by Courant-Fischer) $\vec{y}$ is a non-negative eigenvector of M_G of eigenvalue μ_1, and hence $\vec{y}$ is strictly positive by our Lemma. This means that $\vec{\varphi}_2$ has no zero entries.

Since $\vec{\varphi}_2$ is orthogonal to $\vec{\varphi}_1$ (which has all positive entries), we know $\vec{\varphi}_2$ has both positive and negative entries. Since G ~~has~~ is connected, there must be some edge $a \leftrightarrow b$ with $\varphi_2(a) < 0 < \varphi_2(b)$.

But then

$$\begin{aligned} M_G(a,b)\,\varphi_2(a)\,\varphi_2(b) &= \varphi_2(a)\,\varphi_2(b) \\ &< 0 < |\varphi_2(a)|\,|\varphi_2(b)| = y(a)\,y(b) \\ &= M_G(a,b)\,y(a)\,y(b) \end{aligned}$$

and $\mu_2 < \mu_1$. The contradiction proves that $\mu_2 \neq \mu_1$ (and hence that $\mu_2 < \mu_1$), proving C. $\square$

Now that we have the Perron-Frobenius theorem, we can give a <u>purely spectral</u> characterization of graphs with $\chi(G)=2$ (also called <u>bipartite</u> graphs).

Prop. If G is connected and $\mu_n = -\mu_1$, then G is bipartite.

Proof. In the proof of P-F, we let φ_n be the ~~eigenvalue~~ eigenvector of M_G with eigenvalue μ_n, and considered $\vec{y} = |\vec{\varphi}_n|$, with $\|\vec{\varphi}_n\| = \|\vec{y}\| = 1$. We saw

$$
\begin{aligned}
|\mu_n| &= |\langle \vec{\varphi}_n, M_G \vec{\varphi}_n \rangle| \\
&= \Big| \sum_{a \sim b} \vec{\varphi}_n(a)\, \vec{\varphi}_n(b) \Big| \leq \sum_{a \sim b} |\vec{\varphi}_n(a)|\,|\vec{\varphi}_n(b)| \\
&\leq \sum_{a \sim b} \vec{y}(a)\, \vec{y}(b) = \langle \vec{y}, M_G \vec{y} \rangle \leq \mu_1
\end{aligned}
$$

If all these inequalities are equalities, then all $\vec{\varphi}_n(a)\vec{\varphi}_n(b)$ must have the same sign.

Further, $\vec{y}$ must have Rayleigh quotient μ_1 and hence be a non-negative (=> strictly positive) eigenvector of M_G of eigenvalue μ_1

Since $\vec{y}$ is strictly positive, $\vec{\varphi}_n$ has no zero values and (since $\vec{\varphi}_n$ is orthogonal to $\vec{\varphi}_1$) must have positive and negative values. As before, some edge $a \leftrightarrow b$ must join the positive vertices to the negative ones, and for this edge $\vec{\varphi}_n(a)\vec{\varphi}_n(b) < 0$.

But this means $\vec{\varphi}_n(a)\vec{\varphi}_n(b) < 0$ for <u>every</u> edge $a \bullet\!\!-\!\!\bullet b$. We conclude that

$$\{a \mid \vec{\varphi}_n(a) > 0\} \text{ and } \{a \mid \vec{\varphi}_n(a) < 0\}$$

are a 2-coloring of G. $\square$

We even have a (stronger) converse!

Prop. If G is bipartite, then the eigenvalues $\nu_1 \geq \cdots \geq \nu_n$ are symmetric around zero.

Proof. The definition of bipartite (or 2-colorable) is that $V(G)$ can be partitioned into S and T so that every edge joins a vertex in S to a vertex in T.

Now suppose $\vec{\varphi}$ is an eigenvector of M_G with eigenvalue μ. Define $\vec{x}$ by

$$\vec{x}(a) = \begin{cases} \vec{\varphi}(a), & \text{if } a \in S \\ -\vec{\varphi}(a), & \text{if } a \in T \end{cases}$$

Now if $a \in S$,

$$\begin{aligned}
(M_G \vec{x})(a) &= \sum_{a \leftrightarrow b} M_G(a,b)\, \vec{x}(b) \\
&= \sum_{a \leftrightarrow b} M_G(a,b)\left(-\vec{\varphi}(b)\right) \quad (\text{since } b \in T) \\
&= -\sum_{a \leftrightarrow b} M_G(a,b)\, \vec{\varphi}(b) \\
&= -(M_G \vec{\varphi})(a) \\
&= -\mu\, \vec{\varphi}(a) = -\mu\, \vec{x}(a).
\end{aligned}$$

Similarly, if $a \in T$, $(M_G \vec{x})(a) = -\mu\, \vec{x}(a)$. Thus $\vec{x}$ is an eigenvector of M_G with eigenvalue $-\mu$, completing the proof! $\square$