A (lengthy) example.

We now have defined a lot of stuff! Let's compute it in a particular (gory) example.

Suppose $X(u,v) = (u, v, uv)$. This is the graph of $f(u,v) = uv$, which is a saddle surface.

$$X_u = (1, 0, v) \qquad X_{uu} = (0,0,0)$$

$$X_v = (0, 1, u) \qquad X_{uv} = (0,0,1)$$

$$\qquad\qquad\qquad\qquad\quad \rotatebox{90}{=}$$

$$\qquad\qquad\qquad\qquad X_{vu} = (0,0,1)$$

$$\vec{n} = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(-v, -u, 1)}{\sqrt{1+u^2+v^2}} \qquad X_{vv} = (0,0,0)$$

So

$$E = \langle X_u, X_u \rangle = 1 + v^2 \qquad \ell = \langle \vec{n}, X_{uu} \rangle = 0$$

$$F = \langle X_u, X_v \rangle = uv \qquad m = \langle \vec{n}, X_{uv} \rangle = \frac{1}{\sqrt{1+u^2+v^2}}$$

$$G = \langle X_v, X_v \rangle = 1 + u^2 \qquad n = \langle \vec{n}, X_{vv} \rangle = 0$$

so

$$I_p = \begin{bmatrix} 1+v^2 & uv \\ uv & 1+u^2 \end{bmatrix} \qquad II_p = \frac{1}{\sqrt{1+u^2+v^2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$S_p = I_p^{-1} II_p = \frac{1}{(1+v^2)(1+u^2) - u^2v^2} \begin{bmatrix} 1+u^2 & -uv \\ -uv & 1+v^2 \end{bmatrix} II_p$$

$$= \frac{1}{1+u^2+v^2} \cdot \frac{1}{\sqrt{1+u^2+v^2}} \begin{bmatrix} 1+u^2 & -uv \\ -uv & 1+v^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} -uv & 1+u^2 \\ 1+v^2 & -uv \end{bmatrix}.$$

We now pause to think: how do we calculate the eigenvectors and eigenvalues of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ again?

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we know that the eigenvalues are the roots of

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad-bc)$$

$$= \lambda^2 - \mathrm{tr}(A)\lambda + \det(A),$$

so the solutions are

$$\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}}{2}$$

Now notice that we can already compute

$$K = \det S_p = \frac{1}{(1+u^2+v^2)^3}\left(u^2v^2 - (1+u^2)(1+v^2)\right)$$

$$= \frac{1}{(1+u^2+v^2)^3}\left(- 1 - u^2 - v^2\right)$$

$$= -\frac{1}{(1+u^2+v^2)^2}$$

and

$$H = \frac{1}{2}\operatorname{tr} S_p = -\frac{uv}{\cancel{2}(1+u^2+v^2)^{3/2}}$$

from the matrix. But plugging and chugging, we get principal curvatures

$$K_{1,2} = H \pm \sqrt{H^2 - K}$$

$$= \frac{-uv}{(1+u^2+v^2)^{3/2}} \pm \sqrt{\frac{(-uv)^2}{(1+u^2+v^2)^3} - \frac{-1}{(1+u^2+v^2)^2}}$$

$$= \frac{-uv}{(1+u^2+v^2)^{3/2}} \pm \sqrt{\frac{u^2v^2 + (1+u^2+v^2)}{(1+u^2+v^2)^3}}$$

$$= \frac{-uv \pm \sqrt{(1+u^2)(1+v^2)}}{(1+u^2+v^2)^{3/2}}.$$

Pause again: how do we get the eigen<u>vectors</u>? Recall that the Cayley-Hamilton theorem says that

$$(A-\lambda_1 I)(A-\lambda_2 I) = 0,$$

so both columns of $A-\lambda_2 I$ are in the null space of $A-\lambda_1 I$ (and viceversa). Thus computing a column of $A-\lambda_1 I$ gives you a multiple of v_2 (and vice versa)

$$S_p - \lambda_1 I = \frac{1}{(1+u^2+v^2)^{3/2}} \left(\begin{bmatrix} -uv & 1+u^2 \\ 1+v^2 & -uv \end{bmatrix} - \begin{bmatrix} -uv+\sqrt{(1+u^2)(1+\cdots} & \\ 0 & \end{bmatrix} \right.$$

$$= \frac{1}{(1+u^2+v^2)^{3/2}} \begin{bmatrix} \ast - \sqrt{(1+u^2)(1+v^2)} & 1+u^2 \\ 1+v^2 & -\sqrt{(1+u^2)(1+v^2)} \end{bmatrix}$$

Now either column will do, so pick

$$\frac{1}{(1+u^2+v^2)^{3/2}}\begin{bmatrix} -\sqrt{1+u^2}\sqrt{1+v^2} \\ \sqrt{1+v^2}\sqrt{1+v^2} \end{bmatrix}$$

Since only the ratio of entries matters, we can cancel a common $\sqrt{1+v^2}$ and ~~get~~ the scalar and multiply by -1 to get $\sqrt{1+u^2}\, X_u - \sqrt{1+v^2}\, X_v = V_2$

Similarly, we get

$$V_1 = \sqrt{1+u^2}\, X_u + \sqrt{1+v^2}\, X_v$$

Now we know that ~~this~~ every surface has lines of curvature with tangent vectors in the V_1, V_2 directions

In the u-v plane, the differential equation corresponding to this is

$$\frac{dv}{du} = \pm\frac{\sqrt{1+v^2}}{\sqrt{1+u^2}}$$

Separating variables, that's

$$\int \frac{1}{\sqrt{1+v^2}}\,dv = \pm \int \frac{1}{\sqrt{1+u^2}}\,du$$

We've seen this integral before:

$$\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}},$$

So the solution is

$$\operatorname{arcsinh} v = \pm \operatorname{arcsinh} u + C$$

or

$$v = \sinh(\pm\operatorname{arcsinh} u + C).$$

Now the addition formula for $\sinh x$

is

plus!

$$\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b.$$

so we have

$$v = \pm(\cosh c)\,u + (\sinh c)\cosh(\operatorname{arcsinh} u).$$

We can handle this by recalling

$$\cosh^2 x - \sinh^2 x = 1,$$

so

$$\cosh^{\cancel{2}}(\operatorname{arcsinh} u) = \sqrt{1 + \sinh^2(\operatorname{arcsinh} u)}$$
$$= \sqrt{1+u^2}$$

and finally,

$$v = \pm(\cosh c)\,u + (\sinh c)\sqrt{1+u^2}$$

When $c=0$, we get $v = \pm u$. For other c, we get hyperbolae.

claim we

Pause yet again: How do we recognize an (arbitrary) hyperbola?

Given a general quadratic

$$A \cdot x^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$$

we look at the first part - now revealed as a quadratic form

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle$$

If $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$ has

- eigenvalues of the same sign.
  form is definite (positive or negative)
  quadratic is an <u>ellipse</u>
  we call the ~~form~~ operator <u>elliptic</u>
- one zero eigenvalue
  form is semidefinite
  quadratic is a <u>parabola</u>
  we call the ~~fo~~ operator <u>parabolic</u>

so

$$v \mp (\cosh c)\, u = \sinh c \sqrt{1+u^2}$$

and (squaring both sides)

$$v^2 \mp 2(\cosh c)\, uv + (\cosh^2 c)\, u^2 = \sinh^2 c + (\sinh^2 c)\, u^2$$

using $\cosh^2 x - \sinh^2 x = 1$ again,

$$v^2 \mp 2(\cosh c)\, uv + u^2 - \sinh^2 c = 0$$

and the matrix of quadratic coeffs has determinant

$$\det \begin{bmatrix} 1 & \mp \cosh c \\ \mp \cosh c & 1 \end{bmatrix} = 1 - \cosh^2 c = -\sinh^2 c \leq 0.$$

But $\sinh c = 0 \iff c = 0$, so this is a hyperbola (or the lines $v = \pm u$).

Done!