NOTES

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A Four Vertex Theorem for Polygons

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The classical 4-vertex theorem states that a closed smooth convex plane curve has at least 4 curvature extrema. Published in 1909 **[4]** this theorem and its various generalizations continue to attract interest; see **[1]** for a contemporary point of view on the subject.

One can formulate the 4-vertex theorem in terms of the 1-parameter family of normal lines to the curve; the envelope of these lines is called the *caustic* (or the *evolute*) of the curve. The caustic is not smooth: generically, it has semi-cubical cusp singularities, and these cusps correspond to the extrema of the curvature. One gives the smooth pieces of the caustic the orientation induced by the inward orientation of the normals. As one traverses the caustic, the cusps are the points at which the orientation changes to the opposite; see Figure 1. The next two results, due to O. Musin and B. Wegner, respectively, extend the 4-vertex theorem from smooth curves to polygons. Let $A_1, \dots, A_n$ be consecutive vertices of a plane convex polygon P. Consider the circle that is circumscribed around the triangle $A_{i-1}A_iA_{i+1}$, and let R_i be its radius. Assume, in addition, that for every i the center of the circumscribed circle lies inside the angle $A_{i-1}A_iA_{i+1}$. Likewise, let r_i be the radius of the circle touching the lines $A_{i-1}A_i$, A_iA_{i+1} and $A_{i+1}A_{i+2}$ and lying on the same sides of these lines as P; in other words, the center of this circle is the intersection of the bisectors of the i-th and $(i+1)$-st interior angles of P (here and elsewhere we understand the index i cyclically, that is, $n + 1 = 1$, etc.). Then each of the two cyclic difference sequences

$$(\Delta R)_i = R_{i+1} - R_i \quad \text{and} \quad (\Delta r)_i = r_{i+1} - r_i, \quad i = 1, \dots, n$$

changes sign at least 4 times; see **[7]**, **[8]**, and **[11]**. In the limit $n \to \infty$, when P approximates a smooth curve, both sequences R_i and r_i approximate the radius of

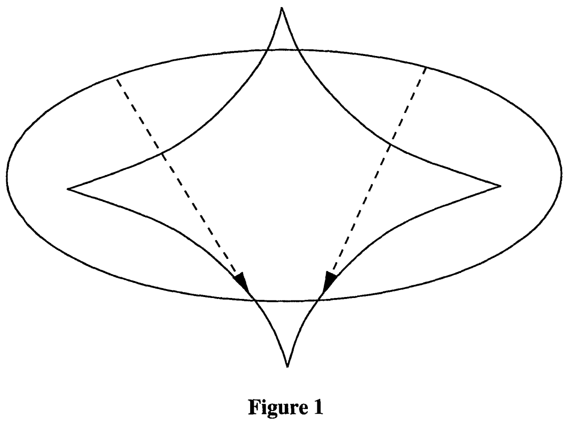

Figure 1

curvature function, and the places where $(\Delta R)_i$ or $(\Delta r)_i$ change signs become critical points of this function, i.e., vertices of the curve.

We propose the following generalization. Consider a convex polygon P with vertices $A_1, \dots, A_n$, and let l_i, $i = 1, \dots, n$ be a line through the vertex A_i that intersect the interior of P and is oriented inwards. Denote by α_i and β_i the angles made by l_i with the sides $A_i A_{i+1}$ and $A_i A_{i-1}$, respectively, and assume that $\alpha_i + \beta_{i+1} < \pi$ for all i.

We think of the polygon P as an analog of a smooth curve, so the lines l_i play the role of the normals. Let $B_i = l_i \cap l_{i+1}$; the closed (possibly self-intersecting) polygon $Q = (B_1, \dots, B_n)$ is the analog of the caustic. Each side of Q lies on a line l_i and gets orientation from it. Traverse Q and mark the vertices at which the orientations of the sides change; by analogy with the smooth case, call them *cusp vertices*; the four cusp vertices are marked in Figure 2. Of course, the number of cusp vertices is even.

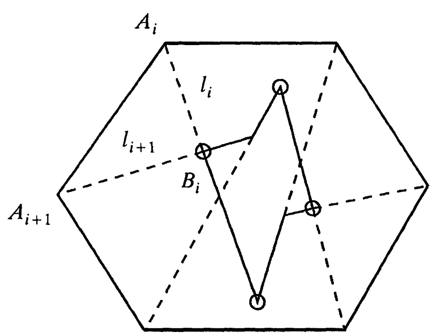

Figure 2

Call a collection of lines l_i *exact* if $\Pi_1^n \sin\alpha_i = \Pi_1^n \sin\beta_i$. For example, the collection of the bisectors of the interior angles of P is exact. A collection of lines l_i is called *generic* if no three consecutive lines intersect at one point.

Theorem 1. *For every exact generic collection of lines $l_1, \dots, l_n$, the polygon Q has at least 4 cusp vertices.*

If l_i are the bisectors of the angles of P, one obtains Wegner's theorem. Note that without the exactness assumption one can easily do without cusp vertices at all; see Figure 3.

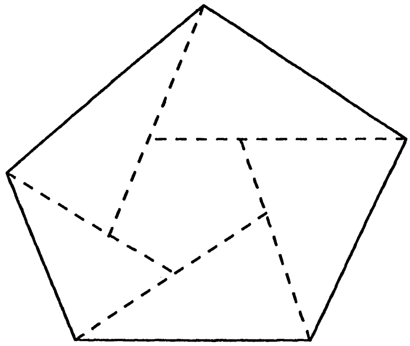

Figure 3

Proof: Start with a geometrical interpretation of the exactness condition. Pick a point $X_1 \in l_1$, draw a line through X_1 parallel to A_1A_2 until it intersects l_2 at point X_2, draw a line through X_2 parallel to A_2A_3 until it intersects l_3 at point $X_3, \ldots$; see Figure 4.

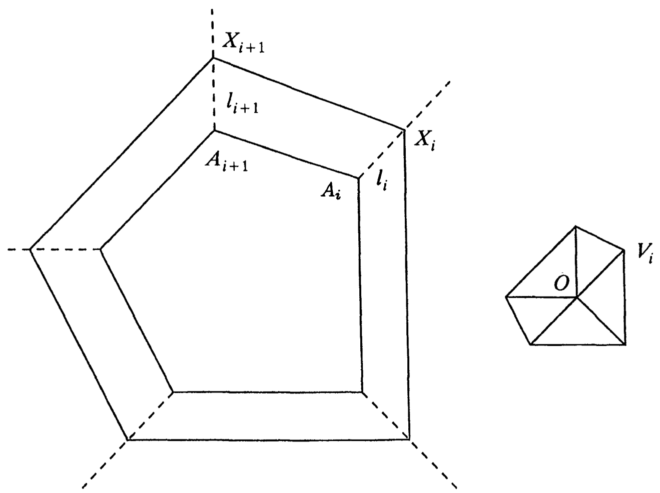

Figure 4

We claim that the polygonal line $X_1, X_2, \ldots$ closes (that is, $X_{n+1} = X_1$) if and only if the collection of lines l_i is exact. Indeed, one has: $|X_{i+1}A_{i+1}|/|X_iA_i| = \sin\alpha_i/\sin\beta_{i+1}$; therefore $X_{n+1} = X_1$ if and only if $\Pi\sin\alpha_i/\Pi\sin\beta_i = 1$.

Translate the vectors A_iX_i to an origin O to obtain a polygon $V_1 \cdots V_n$; see Figure 4. It follows from the argument in the previous paragraph that the sides of this polygon are parallel to those of P. For every i the triangle $B_iA_iA_{i+1}$ is similar to OV_iV_{i+1}; denote by t_i the similarity coefficient. As long as the t_i monotonically increase or decrease, the orientation of the sides of the "caustic" Q do not change, and the cusp vertices of Q correspond to the change of sign of the difference sequence $(\Delta t)_i = t_{i+1} - t_i$; more precisely, B_i is a cusp vertex if and only if $(\Delta t)_{i-1}$ and $(\Delta t)_i$ have opposite signs.

It remains to show that the cyclic difference sequence $(\Delta t)_i$ changes sign at least 4 times. Since $t_i(A_{i+1} - A_i) = V_{i+1} - V_i$, we have

$$\sum_1^n (\Delta t)_i A_{i+1} = -\sum_1^n t_i(A_{i+1} - A_i) = -\sum_1^n (V_{i+1} - V_i) = 0,$$

(the first equality is "integration by parts"). Since $\Sigma(\Delta t)_i = 0$, the cyclic sequence $(\Delta t)_i$ changes sign at least twice. Assume that it happens exactly twice, say, $(\Delta t)_i > 0$ for $i = 1, \ldots, k$ and $(\Delta t)_i < 0$ for $i = k+1, \ldots, n$. Choose an origin on a line m that intersects the segments $A_{k+1}A_{k+2}$ and A_1A_2. With this choice, all of the vectors $(\Delta t)_iA_{i+1}$ lie on the same side of m, and $\Sigma_1^n(\Delta t)_iA_{i+1}$ cannot vanish. This contradiction proves the theorem. ■

Remarks. 1. The continuous case $n \to \infty$ of Theorem 1 is discussed in **[9]** and **[10]**. For another aspect of exact systems of lines see **[6]**.

2. The argument in the proof of Theorem 1 is a particular case of a discrete version of the following theorem (Sturm, Hurwitz, Kellogg, ...): if a function on a circle is L_2-orthogonal to a k-dimensional Chebyshev system, then this function

has at least $k + 1$ distinct zeroes; see [1] for a continuous and [3] for a discrete version.

3. One may argue that the oldest 4-vertex type result is the following Cauchy lemma (1813), which plays a crucial role in the proof of convex polyhedra rigidity: given two convex n-gons whose i-th sides have equal lengths for all i, the cyclic sequence of the differences of the corresponding angles of the polygons changes sign at least 4 times; see [2]. Our theorem concerns a dual configuration: two polygons whose corresponding angles are equal; see Figure 4.

4. A general approach to discrete versions of various 4 and 6-vertex theorems can be found in [5].

We conclude with a curious and not immediately obvious property of exact systems of lines.

Lemma . *Exactness is invariant under projective transformations of the plane.*

Proof: Choose a point O inside the polygon P, let l_i' be the line A_iO and denote by α_i' and β_i' the angles made by l_i' with sides A_iA_{i+1} and A_iA_{i-1}, respectively. The new lines l_i' form an exact system: consider the Sine Rule for the triangle $A_iA_{i+1}O$ and take the product over all i. The cross-ratio of the four lines through vertex A_i, namely, the two adjacent sides of P and the lines l_i and l_i', equals $(\sin \alpha_i'/\sin \alpha_i)/ (\sin \beta_i'/\sin \beta_i)$, and the product of these ratios is equal to one.

Apply a projective transformation F to this configuration of lines and points. The lines $F(l_i')$ are still exact since they pass through a point $F(O)$. The cross-ratios of each quadruple of lines remains the same, and so their product still equals one. It follows that $F(l_i)$ form an exact collection of lines as well. ∎

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