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A GEOMETRIC INEQUALITY FOR PLANE CURVES WITH RESTRICTED CURVATURE

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ABSTRACT. A geometric proof is given that a closed plane curve of length L and curvature bounded by K can be contained inside a circle of radius $L/4 - (\pi - 2)/2K$.

Let K be a positive constant and let E_n be n-dimensional Euclidean space. A continuously differentiable curve X in E_n parametrized by arc length s is called a K-curve if and only if $\|X'(s_1) - X'(s_2)\| \leqslant K|s_1 - s_2|$ for all s_1 and s_2. The purpose of this note is to give a geometric proof of an inequality obtained previously by calculus of variations methods [4]: namely, that if X is a closed K-curve in E_2 of length L, then X lies in a circle of radius R where

$$R \leqslant L/4 - (\pi - 2)/2K. \tag{1}$$

Since the components of X' are functions of bounded variation, $X''(s)$ and $k(s) = \|X''(s)\|$ exist for almost all s and, when $k(s)$ exists, $k(s) \leqslant K$. Thus, K-curves are a generalization of C^2 curves with curvature bounded by K. They share many of the geometrical properties of the latter but are to be preferred in several respects. Dubins [3] showed that among K-curves with prescribed initial and terminal points and prescribed initial and terminal tangent vectors there exists a K-curve of minimal length. We show below (Proposition 3) that the convex envelope of a closed K-curve in E_2 is also a closed K-curve. Both of these properties fail if K-curves are replaced by C^2 K-curves. In fact, Proposition 3 fails if K-curves are replaced by piecewise C^2 K-curves.

To prove inequality (1) we first generalize a theorem of Blaschke [1, p. 116] to the case of convex K-curves, then apply a geometrical construction to show that (1) holds for convex K-curves, and then extend (1) to all K-curves by Proposition 3. An alternative geometric proof of (1) may be obtained by combining results of Dubins [3, Proposition 1 and Theorem 1] with Theorem 2 of Johnson [4]. (We are indebted to the referee for alerting us to Dubins' interesting work and to the existence of the alternative proof.) For information on related problems, consult [2]–[4].

Let C be a closed convex K-curve in E_2 with arc length parameter s. (The arc length parameter of a closed K-curve is understood to assume all real values by periodic extension. Also, a closed K-curve in E_2 is convex if and only

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if it is simple and its inside is a convex set.) Let P be a point of C. With no loss of generality we suppose that $C(0) = P = (0,0)$, that $C'(0) = (1,0)$, and that C lies above the x-axis.

Let $\tau(s)$ be the unique continuous function such that $\tau(0) = 0$ and $C'(s) = (\cos \tau(s), \sin \tau(s))$ for all s. Since C is convex, τ is a nondecreasing function of s. Since C is a K-curve,

$$4 \sin^2 [(\tau(s_1) - \tau(s_2))/2] = \|C'(s_1) - C'(s_2)\|^2 \leqslant K^2 |s_1 - s_2|^2$$

for all s_1 and s_2. With $\varepsilon > 0$ fixed, it follows that $|\tau(s_1) - \tau(s_2)| \leqslant (K + \varepsilon) \cdot |s_1 - s_2|$ for $|s_1 - s_2|$ sufficiently small. By the triangle inequality the restriction on s_1 and s_2 can be removed and, since ε is arbitrary, the inequality $|\tau(s_1) - \tau(s_2)| \leqslant K|s_1 - s_2|$ holds for all s_1 and s_2.

We also have that

$$C(s) = \left(\int_0^s \cos \tau(t)\, dt, \int_0^s \sin \tau(t)\, dt\right)$$

and

$$N(s) = (\sin \tau(s), -\cos \tau(s))$$

where $N(s)$ is the outer-directed normal to C. Thus, the support function $h(s)$, representing the perpendicular distance from $C(0)$ to the tangent line through $C(s)$, is given by

$$h(s) = N(s) \cdot C(s) = \int_0^s \sin(\tau(s) - \tau(t))\, dt$$

(cf. [1]).

Let Γ be a circle of radius $1/K$ tangent to C at P and lying on the same side of the tangent line as C. If the circle Γ is parametrized by arc length σ with $\Gamma(0) = P = (0,0)$ and $\Gamma'(0) = (1,0)$, then $\Gamma'(\sigma) = (\cos K\sigma, \sin K\sigma)$ for all σ, Γ has a support function $h_0(\sigma)$ analogous to that of C, and $h_0(\sigma) = (1 - \cos K\sigma)/K$ by an elementary computation.

Suppose Γ crosses C. By the Mean Value Theorem there will exist points $\Gamma(\sigma)$ and $C(s)$ where the tangent vectors $\Gamma'(\sigma)$ and $C'(s)$ are the same and $h_0(\sigma)$ is larger than $h(s)$. The parameter values σ and s may be chosen to satisfy $K\sigma = \tau(s)$ and without loss of generality we may assume that $0 < K\sigma = \tau(s) \leqslant \pi$. Then σ and s are positive and

$$\begin{aligned} h_0(\sigma) &= \frac{1}{K}\int_0^{\tau(s)} \sin(\tau(s) - \tau)\, d\tau \\ &= \lim_{\text{mesh}\to 0} \sum_{i=1}^{n} \sin(\tau(s) - \tau(s_i))(\tau(s_i) - \tau(s_{i-1}))/K \\ &\leqslant \lim_{\text{mesh}\to 0} \sum_{i=1}^{n} \sin(\tau(s) - \tau(s_i))(s_i - s_{i-1}) \\ &= \int_0^s \sin(\tau(s) - \tau(t))\, dt = h(s). \end{aligned}$$

Here $s_0 \leqslant \cdots \leqslant s_n$ is a partition of $[0,s]$ and $\tau(s_0) \leqslant \cdots \leqslant \tau(s_n)$ is the corresponding partition of $[0, \tau(s)]$. Our conclusion is that $h_0(\sigma) \leqslant h(s)$ contrary to hypothesis, and so we have proved

PROPOSITION 1 (BLASCHKE). *Let C be a closed convex K-curve in E_2. Then through each point P of C passes a circle Γ of radius $1/K$ tangent to C at P and lying inside C.*

Now, if C has length L, let $A = C(s_1)$ and $B = C(s_1 + L/2)$ be two points where $C'(s_1) = -C'(s_1 + L/2)$. Such points exist by a continuity argument. Let M be the midpoint of the line segment AB and let P be an arbitrary point of C.

A and B divide C into two arcs, one of which contains P. Reflect the arc containing P centrally through M to obtain a new curve C^* (see Figure 1) of the same length L but now symmetric about M.

Like C, C^* is a closed convex K-curve. Hence, by Proposition 1, there is a circle Γ of radius $1/K$ tangent to C^* at P and lying inside C^*. Its central reflection Γ_0 through M also lies inside C^*. Likewise, the convex envelope Γ^* of $\Gamma \cup \Gamma_0$ lies inside C^*. Let P_0 be the center of Γ. Then the length of Γ^* is $2\pi/K + 4d(M, P_0)$ and is less than or equal to L. But

$$d(M,P) \leqslant d(M,P_0) + d(P_0,P) = d(M,P_0) + 1/K.$$

Hence, $d(M,P) \leqslant L/4 - (\pi - 2)/2K$. Since P was an arbitrary point of C, we conclude that C lies inside a circle of radius $\leqslant L/4 - (\pi - 2)/2K$ centered at M.

If $d(M,P) = L/4 - (\pi - 2)/2K$ for some point P as above, then the inequalities become equalities and it follows that M, P_0, and P are collinear in that order and $\Gamma^* = C^*$. In such a case C^* is a "racetrack" curve, that is, C^* is the convex envelope of two circles of radius $1/K$ whose centers are a distance $L/2 - \pi/K$ apart. The arc APB of the original curve C is half of C^* and thus is half of a racetrack curve. If a point Q on the arc of C opposite to APB can be found with $d(M,Q) = L/4 - (\pi - 2)/2K$, then AQB is also half of a racetrack curve and $C = APB \cup AQB$ is a racetrack curve. If no such point Q can be found, then $d(M,A) = d(M,B) < L/4 - (\pi - 2)/2K$ and the center M can be shifted toward P so as to obtain a circle of radius $< L/4 - (\pi - 2)/2K$ containing C.

We have proved

PROPOSITION 2. *Let C be a closed convex K-curve in E_2 of length L. Then C lies inside a circle of radius $L/4 - (\pi - 2)/2K$. The curve C may be contained in no smaller circle precisely when C is a racetrack curve of length L and radii $1/K$.*

Let X be a closed K-curve and let C be the convex envelope of X. Observe that every point of C is either a point of X or else an interior point of a line segment in C whose endpoints lie on X. At the latter points, of course, C has its tangent line parallel to the line segment. At points of $C \cap X$ there is likewise a unique supporting line of C: for, if the supporting lines formed a cone at such a point, X could not have a derivative there.

Let σ be an arc length parameter for X and let s be an arc length parameter for C. By what has just been said, $C'(s)$ is a well-defined tangent vector and at points of $C \cap X$ coincides up to sign with $X'(\sigma)$.

Let ε be a positive number and s_0 a particular value of s. We shall show that for s sufficiently close to s_0,

$$\|C'(s) - C'(s_0)\| \leqslant (K + \varepsilon)|s - s_0|.$$

Suppose to the contrary that there exists a sequence $\{s_n\}$ convergent to s_0 such that $\|C'(s_n) = C'(s_0)\| > (K + \varepsilon)|s_n - s_0|$ for all n. If $C(s_0)$ is not on X, then for n large $C(s_n)$ is on the line segment of C through $C(s_0)$. Thence, $C'(s_n)$ equals $C'(s_0)$ for a contradiction. So, $C(s_0)$ must belong to $C \cap X$. We can suppose too that, for each n, $C(s_n)$ belongs to $C \cap X$. Otherwise, the points $C(s_n)$ would be interior points of line segments of C. Without affecting $C'(s_n)$ or increasing $|s_n - s_0|$, we could then shift the points along the segments until they met X.

The curve X has only a finite number of branches which go through $C(s_0)$. By passing to a subsequence, we can suppose that the points $C(s_n)$ all lie on the same branch of X. Thus, there exists a sequence $\{\sigma_n\}$ converging to a parameter value σ_0 with $X(\sigma_0) = C(s_0)$ and $X(\sigma_n) = C(s_n)$ for all n. By passing to subsequences once more, we can assume that, for all n, $C'(s_n) = uX'(\sigma_n)$ where $u = \pm 1$ is fixed. But $\{X'(\sigma_n)\}$ converges to $X'(\sigma_0)$. Hence, $\{C'(s_n)\}$ converges to a limit which up to sign equals $X'(\sigma_0)$ and, therefore, up to sign equals $C'(s_0)$. Since C is a closed convex curve, it cannot reverse direction abruptly. So $\{C'(s_n)\}$ converges to $C'(s_0)$ and $C'(s_0) = uX'(\sigma_0)$.

Then,

$$\begin{aligned}(K + \varepsilon)|s_n - s_0| < \|C'(s_n) - C'(s_0)\| &= \|u(X'(\sigma_n) - X'(\sigma_0))\| \\ &= \|X'(\sigma_n) - X'(\sigma_0)\| \leqslant K|\sigma_n - \sigma_0|.\end{aligned}$$

But for any positive number $\alpha < 1$ and for n sufficiently large,

$$\|(X(\sigma_n) - X(\sigma_0))/(\sigma_n - \sigma_0)\| > 1 - \alpha.$$

Hence,

$$\begin{aligned}(K + \varepsilon)|s_n - s_0| < K|\sigma_n - \sigma_0| &< K\|X(\sigma_n) - X(\sigma_0)\|/(1-\alpha) \\ &= K\|C(s_n) - C(s_0)\|/1 - \alpha \leqslant K|s_n - s_0|/(1-\alpha)\end{aligned}$$

for such n. Thus, $K + \varepsilon < K/(1 - \alpha)$ and, letting α vary, we conclude that $K + \varepsilon \leqslant K$ for a contradiction.

We have established that for each value s_0 and for each s sufficiently close to s_0, $\|C'(s) - C'(s_0)\| \leqslant (K + \varepsilon)|s - s_0|$. By a compactness argument plus the triangle inequality, it follows that C is a $(K + \varepsilon)$-curve. Since this is true for an arbitrary positive number ε, C is a K-curve.

PROPOSITION 3. *Let X be a closed K-curve in E_2. Then the convex envelope C of X is a closed convex K-curve.*

Since the length of C is at most that of X, we have our main result by combining Propositions 2 and 3.

THEOREM 1. *Let X be a closed K-curve in E_2 of length L. Then X lies inside a circle of radius $L/4 - (\pi - 2)/2K$. The curve X may be contained in no smaller circle precisely when X is a racetrack curve of length L and radii $1/K$.*

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