EXISTENCE OF NONTRIVIAL HIGH DISTORTION PAIRS IN A CLASS OF POLYGONAL KNOTS

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1. Introduction

In a 1983 paper Gromov defined the distortion of a curve as follows:

Definition 1.1 ([?gromov]). Let $\gamma : [a, b] \to \mathbb{R}^3$ be a continuous curve. Then the **distortion** of γ is

$$\delta(\gamma) := \sup_{s \neq t} \frac{d(\gamma(s), \gamma(t); \gamma)}{|\gamma(s) - \gamma(t)|}$$

where $d(\gamma(s), \gamma(t); \gamma)$ is the shortest distance from $\gamma(s)$ to $\gamma(t)$ along γ, and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^3$.

Following Mullikin [?Mul], we will denote the quotient inside the supremum above by $dq_\gamma(s, t)$ and call this quantity the *distortion quotient* of (s, t). Furthermore, we define the distortion of a knot:

Definition 1.2. [?Mul] For any knot class $[\gamma]$, define the distortion of the knot class by

$$\delta([\gamma]) := \inf_{\gamma \in [\gamma]} \delta(\gamma).$$

Gromov also asked whether any knot class has distortion less than 100 [?gromov]. More generally, the question that geometric knot theorists want to answer is does there exist an upper bound, C, such that for all knot classes $[K]$, $\delta([K]) < C$? This question has remained unanswered, but there has been work done to try to find lower bounds on the distortion of knots. It is known that the distortion of the unknot is $\pi/2$ [?KS]. Work by Denne and Sullivan shows that the distortion of any nontrivial knot is greater than or equal to $5\pi/3$, and this is the best current lower bound for a nontrivial knot [?DS]. Recently, Chad Mullikin proved that for any knot type, if length minimizing curves in a certain subset of the knot class exist, then these curves have chords with high distortion [?Mul]. Unfortunately, there is no proof for the existence of these minimizing curves. To help strengthen his theorem, Mulikin conjectures that for any curve in a knot class there exists a chord with distinct endpoints such that the distortion quotient of these points is greater than or equal to the distortion of the knot class. The rest of this paper is devoted to answering a variation of this conjecture for polygonal knots.

The distortion of a polygonal knot is just the same as the distortion of any other type of knot, but we need to make a distinction between different pairs of points on the polygon. We will call pairs of points on adjacent edges *trivial* and pairs of points on nonadjacent edges *nontrivial*.

We need to define the *turning angle* at a vertex v_n of a polygon as the angle $\alpha \in [0, \pi]$ between the oriented tangent vectors at v_n to the two edges $v_{n-1}v_n$ and v_nv_{n+1} [?Sul]. Furthermore, we'll call the sum of all the turning angles of a polygon the polygon's total curvature. And if the total curvature of a polygon, P, is finite then we say that the P is in the class of curves of finite total curvature(abbreviated by FTC) [?Sul]. It must also be noted that the distortion at a vertex v with turning angle α_v has distortion

$$\delta(v) = \sec\frac{\alpha_v}{2}$$

[?Sul].

We require a special class of polygons for the theorem:

Definition 1.3. For a given knot class $[\gamma]$, let $\Phi_{[\gamma]}$ be the set of all polygons, $P : [a, b] \to \mathbb{R}^3$, satisfying the following:

(1) P is also an element of $[\gamma]$ and of FTC.
(2) Every pair of adjacent edges of P has combined length strictly less than $L_P/2$, where L_P is the length of P.

The main theorem will say that any polgyon $P \in \Phi_{[\gamma]}$ will have a nontrivial pair of points with distortion greater than or equal to $\delta([\gamma])$.

2. Polygonal Knots

Three lemmas are needed before the proof of the main theorem. The first lemma shows that for $P \in \Phi_{[\gamma]}$, the arclength, $d(\cdot,\cdot;P)$, of two points on adjacent edges is always measured through the common vertex. This lemma is essential in the proof of the theorem. The second lemma shows that a polygonal knot of finite total curvature must have at most a finite number of vertices with distortion greater than or equal to the distortion of the knot class. Finally, the third lemma demonstrates that polygonal knots with at least one vertex with distortion strictly greater than the distortion of the knot class, must have a nontrivial pair with high distortion.

Lemma 2.1. *Let $P \in \Phi_{[\gamma]}$. Then for all pairs $(s, t) \in [a, b] \times [a, b]$, where $P(s)$ and $P(t)$ are on adjacent edges, e_1 and e_2, $d(P(s), P(t); P)$ is the distance from $P(s)$ to $P(t)$ through the common vertex of e_1 and e_2.*

Proof. Take any pair of points (s, t) as described in the statement of the lemma, and denote by $|e_1|$ and $|e_2|$, the lengths of edges e_1 and e_2, respectively. Take the distance from $P(s)$ to $P(t)$ through the common vertex and denote this distance by Δ. The greatest possible value of Δ for a given (s, t) is just $|e_1|+|e_2|$. Furthermore, since $P \in \Phi_{[\gamma]}$ we have $|e_1|+|e_2| < L/2$. So, $\Delta < L_P/2$. Now, $d(P(s), P(t); P)$ is equal to $\min(\Delta, L_P - \Delta)$, and since $\Delta < L_P/2$ we have $L_P - \Delta > L_P/2$. Therefore, $d(P(s), P(t); P) = \Delta$. □

Lemma 2.2. *If $P \in$ FTC and P is a polygon in the knot class $[\gamma]$, then P can have only finitely many vertices whose distortion is greater than or equal to $\delta([\gamma])$.*

Proof. Let v be a vertex with turning angle α_v. The distortion at v is equal to $\sec\frac{\alpha_v}{2}$. Suppose for contradiction that there are infinitely many vertices of P with distortion greater than or equal to $\delta([\gamma])$. Since the secant function is increasing on $[0, \frac{\pi}{2})$, at every vertex whose distortion is greater than or equal to $\delta([\gamma])$ its turning angle must be greater than or equal to $2\sec^{-1}\delta([\gamma])$. At these vertices the turning angles are bounded away from zero, which means the sum of the turning angles is infinite. So, P has infinite curvature, contradicting the hypothesis that $P \in$ FTC. Therefore, P can have only finitely many vertices with distortion greater than or equal to $\delta([\gamma])$. □

Lemma 2.3. *Let $P \in \Phi_{[\gamma]}$. If P has a vertex with distortion strictly greater than $\delta([\gamma])$, then there exists points $(s, t) \in [a, b] \times [a, b]$ such that $P(s)$ and $P(t)$ are nontrivial and $dq_P(s, t) \geq \delta([\gamma])$.*

Proof. Take the maximum, over all pairs of adjacent edges, of the sum of the two edge lengths. Denote this length by M. Let $l = L_P/2 - M > 0$. l is strictly greater than zero since $P \in \Phi_{[\gamma]}$ so no two adjacent edges has combined length greater than or equal to $L_P/2$. Take a vertex, v, of P that has distortion $\delta(v)$, where $\delta(v) > \delta([\gamma])$. Now, let $\Delta = \delta(v) - \delta([\gamma])$. Denote the edges incident at v by e_1 and e_2, and without loss of generality let $|e_1| \leq |e_2|$. Take the other endpoint of e_1 and denote it by $P(s)$. Furthermore, find the point on e_2 that is at a distance $|e_1|$ away from v and denote it by $P(t)$. The key to this proof is the application of Lemma **??**. It tells us that $d(P(s), P(t); P)$ is taken through the vertex v and not around the polygon in the other direction. Therefore, $dq_P(s,t) = \delta(v)$ and by continuity of distortion we can move off of $P(s)$ by a small amount and not change the distortion by more than Δ, as the rest of the proof shows.

Let $a = d(P(s), P(t); P)$, and $b = |P(s) - P(t)|$. We claim that if one takes a point $P(s')$ off of e_1 but at a distance from $P(s)$ less than or equal to

$$\frac{\Delta \cdot b^2}{a - (d+1)b}, \tag{2.1}$$

and less than or equal to l, then $dq_P(s', t) \geq \delta([\gamma])$. To see this, first note that if we move ϵ away from $P(s)$ in arc length, then the distortion quotient is smallest when the Euclidean distance from $P(s')$ to $P(t)$ also increases by ϵ. So without loss of generality assume $d(P(s'), P(t); P) = a + \epsilon$ and $|P(s') - P(t)| = b + \epsilon$, where ϵ is less than or equal to (2.1) and less than or equal to l. Since we move less than l from $P(s)$, we still calculate distance along P from $P(s')$ to $P(t)$ through v. Now take the difference of the distortion quotients:

$$\begin{aligned} dq_P(s,t) - dq_P(s',t) &= \frac{a}{b} - \frac{a+\epsilon}{b+\epsilon} \\ &= \frac{\epsilon(a-b)}{b(b+\epsilon)} \end{aligned}$$

and,

$$\begin{aligned} \frac{(b+\epsilon)}{\epsilon} \cdot \frac{b}{(a-b)} &= \frac{b^2}{\epsilon(a-b)} + \frac{b}{a-b} \\ &\geq \frac{a-(d+1)b}{db^2} \cdot \frac{b^2}{a-b} + \frac{b}{a-b} \\ &= \frac{a-(\Delta+1)b}{d(a-b)} + \frac{bd}{d(a-b)} \\ &= \frac{1}{\Delta} \end{aligned}$$

Therefore, $dq_P(s,t) - dq_P(s',t) \leq \Delta$, which implies $\delta(v) - dq_P(s',t) \leq \delta(v) - \delta([\gamma])$. Finally, we get $dq_P(s',t) \geq \delta([\gamma])$, and $P(s')$ and $P(t)$ are points on nonadjacent edges. □

Theorem 2.1. *A polygon $Q \in \Phi_{[\gamma]}$, has a nontrivial pair of points, $Q(s)$ and $Q(t)$, such that $dq_Q(s,t) \geq \delta([\gamma])$.*

Proof. Consider the set $\Gamma \subset \Phi_{[\gamma]}$ where every $P \in \Gamma$ satisfies the condition that $\forall s,t \in [a,b], dq_P(s,t) < \delta([\gamma])$ when $P(s)$ and $P(t)$ are on nonadjacent edges. From Lemma 2.3 we see that the distortion at every vertex of $P \in \Gamma$ must be less than or equal to $\delta([\gamma])$. Furthermore, Lemma 2.2 implies that only finitely many of these vertices can have distortion equal to $\delta([\gamma])$. Also, for any $P \in \Gamma$, P must have at least one vertex with distortion equal to $\delta([\gamma])$. For if P did not, then for every $s,t \in [a,b], dq_P(s,t) < \delta([\gamma])$. This implies $\delta(P) < \delta([\gamma])$, which means $P \notin [\gamma]$. But $P \in \Phi_{[\gamma]}$, so we've reached a contradiction.

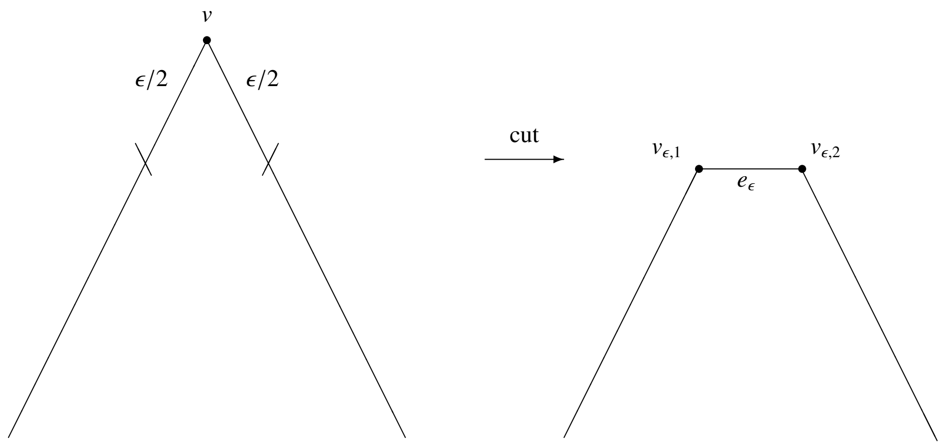

FIGURE 2.1. The polygon cutting process

To prove this theorem we want to show that Γ is empty, and we'll accomplish this by contradiction. So suppose that Γ is not empty. Then there must exist $Q \in \Gamma$ with the property that Q has the least number of vertices with distortion equal to $\delta([\gamma])$. Take a vertex v of Q that has distortion $\delta([\gamma])$. Such a v must exist as we've shown. As noted in the introduction, the turning angle at v, α_v, satisfies the equation:

$$\sec \frac{\alpha_v}{2} = \delta([\gamma]).$$

We now wish to remove a length of $\epsilon/2$ from both sides of v and add a new edge, e_ϵ(see figure **??**). But this process might cause drastic changes in our polygon. First, we must take care that our resulting polygon, which we will denote by Q_ϵ, stays in $\Phi_{[\gamma]}$. We want Q_ϵ to be in $[\gamma]$. To ensure this we need to make ϵ small enough so that no other edge of Q_ϵ passed through the removed "triangle". If ϵ could not be made small enough to ensure no other edges passed between e_ϵ and v, there would be points on the polygon arbitrarily close to v in space but bounded away from zero in arc length. This would result in infinite distortion, which is a clear contradiction of the fact that the distortion of Q is $\delta([\gamma])$. Therefore, we can definitely choose an ϵ small enough so that $Q_\epsilon \in [\gamma]$.

Two new vertices, $v_{\epsilon,1}$ and $v_{\epsilon,2}$, are created in the construction of Q_ϵ, but their turning angles are exactly half of the turning angle of v. Then the total curvature of Q_ϵ is equal to the total curvature of Q, which gives $Q_\epsilon \in FTC$. Therefore, we can choose ϵ small enough so that Q_ϵ satisfies property (1) in the definition of $\Phi_{[\gamma]}$.

It remains to show that ϵ can be choosen to assure Q_ϵ satisfies property (2) in the definition of $\Phi_{[\gamma]}$. An easy calculation shows the length of e_ϵ is $\epsilon/\delta([\gamma])$, so the total length of Q_ϵ, L_{Q_ϵ}, is

$$L_Q - (\epsilon - \epsilon/\delta([\gamma])).$$

Let M be the maximum, over all pairs of adjacent edges in P, of the sum of the two edge lengths. We need the change in arc length to be less than $\frac{L_Q}{2} - M$ because this would ensure that no pair of adjacent edges in Q_ϵ has length more than $L_{Q_\epsilon}/2$. Choosing an $\epsilon > 0$ such that

$$\epsilon < \frac{\delta([\gamma])}{\delta([\gamma]) - 1} \cdot (\frac{L_Q}{2} - M) \tag{2.2}$$

results in the change in arc length to be small enough to preserve property (2) in the definition of $\Phi_{[\gamma]}$. Therefore, choosing ϵ satisfying **??** and small enough so that no edge runs between e_ϵ and v, results in $Q_\epsilon \in \Phi_{[\gamma]}$.

Choose an ϵ so that $Q_\epsilon \in \Phi_{[\gamma]}$. Then cut off $\epsilon/2$ from both sides of v and construct the new edge, e_ϵ. Consider the vertices, $v_{\epsilon,1}$ and $v_{\epsilon,2}$, in Q_ϵ added in the construction above. The turning angles at these vertices are strictly less than the turning angle at v in Q, which implies that the distortion values at $v_{\epsilon,1}$ and $v_{\epsilon,2}$ are both strictly less than $\delta([\gamma])$. Therefore, Q_ϵ, which is still in $\Phi_{[\gamma]}$, has one fewer vertex than Q that realizes $\delta([\gamma])$. So, if $Q_\epsilon \in \Gamma$ we have a contradiction since Q was assumed to have the fewest vertices that realize the distortion.

If $Q_\epsilon \notin \Gamma$, then there exists $s, t \in [a, b]$ such that $Q_\epsilon(s), Q_\epsilon(t)$ are nontrivial and $dq_{Q_\epsilon}(s, t) \geq \delta([\gamma])$. We now want to show that exactly one of the points, $Q_\epsilon(s)$ or $Q_\epsilon(t)$, must be on e_ϵ. Denote the two edges that were incident at v by e_1 and e_2, and the modified edges by e'_1 and e'_2. So $e'_j \subset e_j$ for $j = 1, 2$. Suppose neither $Q_\epsilon(s)$ nor $Q_\epsilon(t)$ are in e_ϵ. Clearly, $d(Q_\epsilon(s), Q_\epsilon(t); Q_\epsilon) \leq d(Q(s), Q(t); Q)$, so $dq_{Q_\epsilon}(s, t) \leq dq_Q(s, t)$ since the distance in $\mathbb{R}^3$ from $P(s)$ to $P(t)$ does not change. It is possible that $dq_Q(s, t) = \delta([\gamma])$ if $Q_\epsilon(s)$ is on either e'_1 or e'_2, and $Q_\epsilon(t)$ is on the other. But in this case we would have $d(Q_\epsilon(s), Q_\epsilon(t); Q_\epsilon) < d(Q(s), Q(t); Q)$ because $Q \in \Phi_{[\gamma]}$ which means the arc length is measured through v. This leads to a contradiction: $\delta([\gamma]) \leq dq_{Q_\epsilon}(s, t) < dq_Q(s, t) = \delta([\gamma])$. So, at least one of the points must be on e_ϵ. They cannot both be on it otherwise $dq_{Q_\epsilon}(s, t) = 1$, but for all closed curves the distortion is greater than or equal to $\pi/2$ [?KS]. Therefore, exactly one of the points,$Q_\epsilon(s)$ and $Q_\epsilon(t)$, is in e_ϵ. Without loss of generality, let $Q_\epsilon(s) \in e_\epsilon$.

Consider the sequence $\{\epsilon_k\}_{k=1}^{\infty}, k \in \mathbb{N}$, such that ϵ_1 is the ϵ choosen above and $0 < \epsilon_{i+1} < \epsilon_i$. For every ϵ_i, $Q_{\epsilon_i} \in \Phi_{[\gamma]}$ since $\epsilon_i \leq \epsilon$, and $Q_{\epsilon_i} \notin \Gamma$ for all i because otherwise there would be a $Q_{\epsilon_j} \in \Gamma$ with one fewer vertex than Q with distortion $\delta([\gamma])$. As shown above, for each Q_{ϵ_i}, there are points $(s_i, t_i) \in [a, b] \times [a, b]$ such that $Q_{\epsilon_i}(s_i) \in e_{\epsilon_i}$, $Q_{\epsilon_i}(t_i)$ is on an edge nonadjacent to e_{ϵ_i}(so $Q_{\epsilon_i}(t_i) \in Q$), and $dq_{Q_{\epsilon_i}}(s_i, t_i) \geq \delta([\gamma])$. By compactness of $[a, b] \times [a, b]$ there must be a subsequence of $\{(s_i, t_i)\}_{i=1}^{\infty}$ converging to (s, t). $Q(s)$ must be the vertex v of Q removed in the construction of the new polygons, and since $Q_{\epsilon_i}(t_i)$ is on an edge nonadjacent to e_{ϵ_i}, $Q(t)$ is bounded away from $Q(s)$.

Let $\{s_i, t_i)\}_{i=1}^{\infty}$ be the convergent subsequence converging to (s, t) as above. To ease the forthcoming computations denote the polygon Q_{ϵ_j} by Q_j. Since $dq_{Q_j}(s_j, t_j) \geq \delta([\gamma])$ for all j, we want to show that for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for every $j \geq N$, $\left|dq_Q(s, t) - dq_{Q_j}(s_j, t_j)\right| \leq \epsilon$. As the following calculations show, we want to bound three terms:

$$\begin{aligned}\left|dq_Q(s,t)-dq_{Q_j}(s_j,t_j)\right| &= \left|dq_Q(s,t)-dq_Q(s,t_j)-dq_{Q_j}(s_j,t_j)+dq_Q(s,t_j)\right|\\ &\le \left|dq_Q(s,t)-dq_Q(s,t_j)\right|+\left|dq_{Q_j}(s_j,t_j)-dq_Q(s,t_j)\right|\\ &\le \left|dq_Q(s,t)-dq_Q(s,t_j)\right|+\left|dq_{Q_j}(s_j,t_j)-dq_{Q_j}(s_j,t_j)\frac{\left|Q_j(s_j)-Q_j(t_j)\right|}{\left|Q(s)-Q_j(t_j)\right|}\right|\\ &+ \left|dq_Q(s,t_j)-dq_{Q_j}(s_j,t_j)\frac{\left|Q_j(s_j)-Q_j(t_j)\right|}{\left|Q(s)-Q_j(t_j)\right|}\right|. \qquad (2.3)\end{aligned}$$

Fix an $\epsilon > 0$. By continuity of dq_Q and by convergence of $t_j \to t$, there exists an $N_1 \in \mathbb{N}$ such that for every $j \ge N_1$, we have $\left|dq_Q(s,t)-dq_Q(s,t_j)\right| \le \frac{\epsilon}{2}$. Furthermore, simplifying the third term in **??** above yields:

$$\left|dq_Q(s,t_j)-dq_{Q_j}(s_j,t_j)\frac{\left|Q_j(s_j)-Q_j(t_j)\right|}{\left|Q(s)-Q_j(t_j)\right|}\right| = \frac{\left|d(s,t_j;Q)-d(s_j,t_j;Q_j)\right|}{\left|Q(s)-Q_j(t_j)\right|}. \qquad (2.4)$$

The above equality holds since $Q(t_j) = Q_j(t_j)$ for all j. $\left|Q(s)-Q_j(t_j)\right|$ is bounded away from 0, which implies there exists a constant $C_1 > 0$ such that for all j, $\frac{1}{\left|Q(s)-Q_j(t_j)\right|} \le C_1$. Furthermore, $\left|d(s,t_j;Q)-d(s_j,t_j;Q_j)\right| \le \left|\frac{\epsilon_j}{2}-\frac{\epsilon_j}{\delta([\gamma])}\right| = \epsilon_j \cdot \left|\frac{\delta([\gamma])-2}{2\delta([\gamma])}\right|$, and since $\epsilon_j \to 0$ as $j \to \infty$, there exists an $N_2 \in \mathbb{N}$ such that $\epsilon_j \le \frac{\epsilon}{4C_1} \cdot \left|\frac{2\delta([\gamma])}{\delta([\gamma])-2}\right|$. So given $j \ge N_2$, equation **??** yields,

$$\begin{aligned}\left|dq_Q(s,t_j)-dq_{Q_j}(s_j,t_j)\frac{\left|Q_j(s_j)-Q_j(t_j)\right|}{\left|Q(s)-Q_j(t_j)\right|}\right| &\le C_1\cdot\epsilon_j\cdot\left|\frac{\delta([\gamma])-2}{2\delta([\gamma])}\right|\\ &\le \frac{\epsilon}{4}.\end{aligned}$$

Now we are left with one term in **??** to bound. To begin we see that there exists a constant $C_2 > 0$ such that for all j, $dq_{Q_j}(s_j,t_j) \le C_2$. $Q_j(s_j)$ converges to $Q(s)$, which means there exists an $N_3 \in \mathbb{N}$ such that for all $j \ge N_3$, $\left|Q(s)-Q_j(s_j)\right| \le \frac{\epsilon}{4C_1C_2}$. From the triangle inequality we get,

$$\left|Q(s)-Q_j(s_j)\right| \ge \left|\left|Q(s)-Q_j(t_j)\right|-\left|Q_j(s_j)-Q_j(t_j)\right|\right|$$

So,

$$-\frac{\epsilon}{4C_1C_2} \le \left|Q(s)-Q_j(t_j)\right|-\left|Q_j(s_j)-Q_j(t_j)\right| \le \frac{\epsilon}{4C_1C_2}.$$

A further simplification yields,

$$1-\frac{\epsilon}{4C_1C_2}\cdot\frac{1}{\left|Q(s)-Q_j(t_j)\right|} \le \frac{\left|Q_j(s_j)-Q_j(t_j)\right|}{\left|Q(s)-Q_j(t_j)\right|} \le 1+\frac{\epsilon}{4C_1C_2}\cdot\frac{1}{\left|Q(s)-Q_j(t_j)\right|}$$

As we've seen, $\frac{1}{\left|Q(s)-Q_j(t_j)\right|} \le C_1$ so

$$1-\frac{\epsilon}{4C_2} \le \frac{\left|Q_j(s_j)-Q_j(t_j)\right|}{\left|Q(s)-Q_j(t_j)\right|} \le 1+\frac{\epsilon}{4C_2}. \qquad (2.5)$$

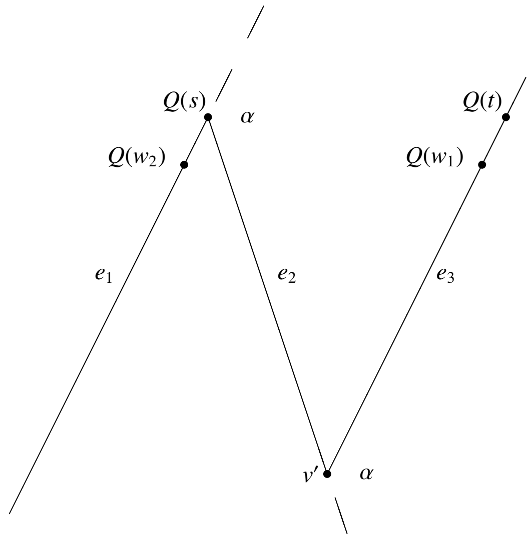

FIGURE 2.2

Multiplying **??** through by $dq_{Q_j}(s_j,t_j)$ and simplifying gives,

$$\begin{aligned}\left|dq_{Q_j}(s_j,t_j)\cdot\frac{\left|Q_j(s_j)-Q_j(t_j)\right|}{\left|Q(s)-Q_j(t_j)\right|}-dq_{Q_j}(s_j,t_j)\right| &\leq \frac{\epsilon}{4C_2}dq_{Q_j}(s_j,t_j)\\ &\leq \frac{\epsilon}{4C_2}\cdot C_2\\ &= \frac{\epsilon}{4}.\end{aligned}$$

Therefore, from **??** and the computed bounds, we conclude that given an $\epsilon>0$ there exists an $N=max(N_1,N_2,N_3)\in\mathbb{N}$ such that for all $j\geq N$,

$$\left|dq_Q(s,t)-dq_{Q_j}(s_j,t_j)\right|\leq\frac{\epsilon}{2}+\frac{\epsilon}{4}+\frac{\epsilon}{4}=\epsilon.$$

Since $dq_{Q_j}(s_j,t_j)\geq\delta([\gamma])$ for all j, we see that $dq_Q(s,t)\geq\delta([\gamma])$. But since $Q\in\Gamma$, $dq_Q(s,t)$ must equal $\delta([\gamma])$ and $Q(s)$ and $Q(t)$ must be on adjacent edges.

So, if v' is the vertex between $Q(s)$ and $Q(t)$, the distances along Q from v' to $Q(s)$ and from v' to $Q(t)$ must be equal. The turning angle at v' is the same as at $v=Q(s)$. Now move a small distance along Q from $Q(t)$ towards v' to a point $Q(w_1)$. Move the same distance from $Q(s)$ away from v' to a point $Q(w_2)$. $dq_Q(w_1,w_2)$ is minimized when e_1,e_2,and e_3 are in the same plane since this configuration maximizes $|Q(w_1)-Q(w_2)|$. See figure **??**. But then e_1 and e_2 are parallel, so $dq_Q(w_1,w_2)=\delta([\gamma])$. This is a contradiction since e_1 and e_3 are nonadjacent, $Q(w_1)$ is not v', which means $Q(w_1)$ and $Q(w_2)$ are nontrivial .

Therefore, Γ must be empty, which means there exists no polygon of a given knot with only nontrivial pairs of points with distortion greater than or equal to the distortion of the knot class.

□

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