

AN INEQUALITY FOR CLOSED SPACE CURVES

G. D. CHAKERIAN

1. Among a number of interesting results in a paper of I. Fáry (see [2]) appears the following. Let C be a rectifiable closed curve of length $L(C)$ and total curvature $\kappa(C)$ enclosed by a sphere S of radius r in Euclidean 3-space. Then

$$(1) \quad L(C) \leq \frac{4}{\pi} r\kappa(C).$$

The proof of (1) rests upon the corresponding inequality for plane closed curves, which states that if C is enclosed by a circle of radius r , then

$$(2) \quad L(C) \leq r\kappa(C).$$

The latter inequality gives a sharp result, with equality obtained in case C is a circle of radius r .

In this paper we sharpen (1) to the following result. Let C be a rectifiable closed curve enclosed by a $k-1$ dimensional sphere S of radius r in Euclidean k -space, $k \geq 2$. Then

$$(3) \quad L(C) \leq r\kappa(C).$$

The proof of (3) again depends on the plane case and is motivated by the following construction. We form the cone T over the curve C with apex at the center of S , slit along a longest generator and develop the result in a plane. The resulting plane arc C' is completed to a closed plane curve C'' by attaching an arc of a circle. It is noted that the curvature of C' is equal pointwise to the geodesic curvature of C with respect to T , which in turn is not greater, pointwise, than the curvature of C . The length of C' is the same as that of C . The inequality (2) applied to C'' now gives (3).

2. In this section we prove some lemmas which lead directly to the main theorem.

LEMMA 1. *Let C be a rectifiable plane arc of length L . For any line G , let $n(p, \theta)$ be the number of intersections of G with C , where (p, θ) , $p \geq 0$, $0 \leq \theta < 2\pi$, are the normal coordinates of G . Then*

$$(4) \quad L = \frac{1}{2} \int_0^{2\pi} \int_0^{\infty} n(p, \theta) dp d\theta.$$

Received March 9, 1961.

This striking formula of Crofton is proved by Blaschke, [1], page 46.

LEMMA 2. Let C be a closed plane curve parametrized by arc length s . Let $\vec{r} = \vec{r}(s)$, $0 \leq s \leq L$, be the tracing vector of, C , and assume \vec{r}'' exists and is continuous except at a finite number of points $\vec{r}(s_1), \dots, \vec{r}(s_m)$, where there are corners with "exterior" angles $\alpha_1, \dots, \alpha_m$ respectively. Given any direction θ , $0 \leq \theta < 2\pi$, let $n(\theta)$ be the number of tangents to C orthogonal to that direction, where a tangent to C at $\vec{r}(s_i)$, $i = 1, 2, \dots, m$, means a line through the point but not crossing C at that point. Then

$$(5) \quad \frac{1}{2} \int_0^{2\pi} n(\theta) d\theta = \int |\vec{r}''(s)| ds + \sum_{i=1}^m \alpha_i = \text{total curvature of } C,$$

where the integral on the right is extended over the smooth part of C .

Proof. We may write $n(\theta) = \sum_{i=0}^m n_i(\theta)$, where $n_0(\theta)$ counts the number of tangents to the smooth part of C and $n_i(\theta)$, $i \neq 0$, counts the number of tangents at $\vec{r}(s_i)$. Clearly n_i takes only the values 0 or 1, for $i \neq 0$, and

$$(6) \quad \frac{1}{2} \int_0^{2\pi} n_i(\theta) d\theta = \alpha_i, \quad i \neq 0.$$

Finally, we have that

$$(7) \quad \frac{1}{2} \int_0^{2\pi} n_0(\theta) d\theta = \int |\vec{r}''(s)| ds,$$

since the left hand side is just the measure of the spherical image (counting multiplicity) of the smooth part of C .

LEMMA 3. Let $\vec{x}_0, \vec{x}_1, \dots, \vec{x}_n$, be the successive vertices of a plane polygon \bar{P} enclosed by a circle S of radius r_0 . Suppose further that the "initial" and "end" points, \vec{x}_0 and \vec{x}_n respectively, lie on S . Let α_i , $0 \leq \alpha_i \leq \pi$, be the angle between $\vec{x}_{i+1} - \vec{x}_i$ and $\vec{x}_i - \vec{x}_{i-1}$, $i = 1, \dots, n-1$. If $\vec{x}_0 \neq \vec{x}_n$, let α_0 , $0 \leq \alpha_0 \leq \pi$, be the angle between $\vec{x}_1 - \vec{x}_0$ and the unit tangent vector to S (with counterclockwise orientation) at \vec{x}_0 , and let α_n , $0 \leq \alpha_n \leq \pi$, be the angle between $\vec{x}_n - \vec{x}_{n-1}$ and the unit tangent vector to S (with counterclockwise orientation) at \vec{x}_n . If $\vec{x}_0 = \vec{x}_n$, then simply let $\alpha_0 (= \alpha_n)$, $0 \leq \alpha_0 \leq \pi$, be the angle between $\vec{x}_1 - \vec{x}_0$ and $\vec{x}_0 - \vec{x}_{n-1}$. Let $L(\bar{P})$ be the length of \bar{P} .

Then if $\vec{x}_0 \neq \vec{x}_n$, we have that

$$(8) \quad L(\bar{P}) \leq r_0 \sum_{i=0}^n \alpha_i.$$

If $\vec{x}_0 = \vec{x}_n$, we have

$$(8') \quad L(\bar{P}) \leq r_0 \sum_{i=0}^{n-1} \alpha_i .$$

(This lemma is a special case of Fáry's theorem for the plane. See [2], page 121. The proof we give here is essentially that of Fáry.)

Proof. We consider first the case where $\vec{x}_0 \neq \vec{x}_n$. Let \bar{S} be the arc of S traversed in a counterclockwise direction in going along S from \vec{x}_n to \vec{x}_0 . Let $C = \bar{P} \cup \bar{S}$. Let δ be the angle subtended at the center of S by \bar{S} . Then Lemma 1 gives,

$$(9) \quad L(\bar{P}) + r_0 \delta = L(C) = \frac{1}{2} \int_0^{2\pi} \int_0^{r_0} n(p, \theta) \, dp d\theta .$$

It is easy to see, however, that $n(p, \theta) \leq n(\theta)$ for $0 \leq \theta < 2\pi$. Hence, by (9) and (5), we have

$$(10) \quad L(\bar{P}) + r_0 \delta \leq \frac{1}{2} r_0 \int_0^{2\pi} n(\theta) \, d\theta = r_0 \left(\sum_{i=0}^n \alpha_i + \delta \right) .$$

This gives the assertion for $\vec{x}_0 \neq \vec{x}_n$. The case $\vec{x}_0 = \vec{x}_n$ is now clear.

LEMMA 4. *Let P be a closed polygon enclosed by a $k - 1$ dimensional sphere S of radius r in Euclidean k -space. Let $\vec{y}_0, \vec{y}_1, \dots, \vec{y}_n = \vec{y}_0$, be the successive vertices of P . Let $\beta_i, 0 \leq \beta_i \leq \pi$, be the angle between $\vec{y}_{i+1} - \vec{y}_i$ and $\vec{y}_i - \vec{y}_{i-1}$, $i = 0, 1, \dots, n - 1$, where \vec{y}_{-1} is defined to be \vec{y}_{n-1} . Define the total curvature, $\kappa(P)$, of P , by*

$$(11) \quad \kappa(P) = \sum_{i=0}^{n-1} \beta_i , \quad (\text{See Milnor, [3], p. 249.})$$

Let $L(P)$ be the length of P . Then

$$(12) \quad L(P) \leq r \kappa(P) .$$

Proof. Let \vec{o} be the center of S . Assume that the vertices of P are labeled so that \vec{y}_0 is no closer to \vec{o} than any other vertex. Let $\beta'_i, 0 \leq \beta'_i \leq \pi$, be the angle between $\vec{y}_i - \vec{o}$ and $\vec{y}_i - \vec{y}_{i+1}$; let $\beta''_i, 0 \leq \beta''_i \leq \pi$, be the angle between $\vec{y}_i - \vec{o}$ and $\vec{y}_i - \vec{y}_{i-1}$, $i = 0, 1, \dots, n - 1$. The triangle inequality applied to a spherical triangle cut out of a sphere centered at \vec{y}_i shows that

$$\beta'_i + \beta''_i \geq \pi - \beta_i, \text{ and } (\pi - \beta'_i) + (\pi - \beta''_i) \geq \pi - \beta_i .$$

Hence,

$$(13) \quad |\pi - (\beta'_i + \beta''_i)| \leq \beta_i , \quad i = 0, 1, \dots, n - 1 .$$

We now form the cone over P with apex at \vec{o} , cut along the edge connecting \vec{o} to \vec{y}_0 and develop the result in a plane as follows. Let \vec{p} be a fixed point in the plane R^2 . We map \vec{y}_0 into any point $\vec{x}_0 \in R^2$ satisfying $|\vec{x}_0 - \vec{p}| = |\vec{y}_0 - \vec{o}| = r_0$. We next map \vec{y}_1 into a point $\vec{x}_1 \in R^2$ satisfying $|\vec{x}_1 - \vec{p}| = |\vec{y}_1 - \vec{o}| = r_1$, and such that the angle δ_1 , from $\vec{x}_0 - \vec{p}$ to $\vec{x}_1 - \vec{p}$, measured in a counterclockwise direction, is equal to the angle δ_1 , $0 \leq \delta_1 \leq \pi$, between $\vec{y}_0 - \vec{o}$ and $\vec{y}_1 - \vec{o}$. In general we map \vec{y}_i into $\vec{x}_i \in R^2$ so that $|\vec{x}_i - \vec{p}| = |\vec{y}_i - \vec{o}| = r_i$ and the angle δ_i from $\vec{x}_{i-1} - \vec{p}$ to $\vec{x}_i - \vec{p}$, measured counterclockwise, is equal to the angle δ_i , $0 \leq \delta_i \leq \pi$, between $\vec{y}_{i-1} - \vec{o}$ and $\vec{y}_i - \vec{o}$. This construction gives us a polygon \bar{P} in R^2 . Construct the circle S' of radius r_0 centered at \vec{p} . Then \bar{P} is enclosed by S' , and \vec{x}_0 and \vec{x}_n (in general $\vec{x}_0 \neq \vec{x}_n$) are on S' . It is easily seen that the angle α_i , $0 \leq \alpha_i \leq \pi$, between $\vec{x}_i - \vec{x}_{i-1}$ and $\vec{x}_{i+1} - \vec{x}_i$, is $|\pi - (\beta'_i + \beta''_i)|$, $i = 1, 2, \dots, n-1$. It is also seen that the angles α_0 and α_n described in Lemma 3 are equal to $(\pi/2) - \beta'_0 > 0$ and $(\pi/2) - \beta''_0 > 0$ respectively if $\vec{x}_0 \neq \vec{x}_n$ and are both equal to $\pi - (\beta'_0 + \beta''_0) > 0$ if $\vec{x}_0 = \vec{x}_n$. Hence if $\vec{x}_0 \neq \vec{x}_n$,

$$(14) \quad \sum_{i=0}^n \alpha_i = \frac{\pi}{2} - \beta'_0 + \sum_{i=1}^{n-1} |\pi - (\beta'_i + \beta''_i)| + \frac{\pi}{2} - \beta''_0 \\ = \sum_{i=0}^{n-1} |\pi - (\beta'_i + \beta''_i)|,$$

and if $\vec{x}_0 = \vec{x}_n$,

$$(14') \quad \sum_{i=0}^{n-1} \alpha_i = \sum_{i=0}^{n-1} |\pi - (\beta'_i + \beta''_i)|.$$

Therefore, by (8), (8'), (14), and (14'),

$$L(P) = L(\bar{P}) \leq r_0 \sum_{i=0}^{n-1} |\pi - (\beta'_i + \beta''_i)| \leq r_0 \sum_{i=0}^{n-1} \beta_i = r_0 \kappa(P) \leq r \kappa(P).$$

3. THEOREM 1. *Let C be a rectifiable closed curve enclosed by a $k-1$ dimensional sphere S of radius r in Euclidean k -space, $k \geq 2$. Let $L(C)$ be the length of C and $\kappa(C)$ be the total curvature of C . ($\kappa(C) = 1.\text{u.b.} \kappa(P)$, where P runs over all polygons inscribed in C . See Milnor, [3].) Then*

$$L(C) \leq r \kappa(C).$$

Proof. Given any $\varepsilon > 0$, there is a polygon P inscribed in C such that $L(C) - L(P) \leq \varepsilon$. We have that $\kappa(P) \leq \kappa(C)$. Hence

$$L(C) - \varepsilon \leq L(P) \leq r \kappa(P) \leq r \kappa(C).$$

The theorem follows.

COROLLARY. *Let C be a closed curve of class C'' enclosed by a unit $k - 1$ dimensional sphere in Euclidean k -space. Let $\kappa(s) = |\vec{r}''(s)| =$ curvature of C at $\vec{r}(s)$, $0 \leq s \leq L(C)$. Then*

$$(15) \quad \max \kappa \geq 1 .$$

Proof.

$$L(C) \leq \kappa(C) = \int_0^{L(C)} \kappa(s) ds \leq \max \kappa \cdot L(C) .$$

Note that we have used the fact that the above integral form for the total curvature coincides with the previous definition. This is proved by Milnor in [3].

REFERENCES

1. W. Blaschke, *Vorlesungen über Integralgeometrie*, Chelsea, 1949.
2. I. Fáry, *Sur certaines inégalités géométriques*, Acta Sci. Math., Szeged, **12** (1950), 117-124.
3. J. W. Milnor, *On the total curvature of knots*, Ann. of Math., **52** (1950), 248-257.

