Case 2 for Theorem 1

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1 Setup

For Case 2, $e_1 \neq e_2$ and so for our "variational" calculations we choose:

$$\xi = e_1 \ominus e_2 \tag{1}$$

We know (or can at least compute easily after the first section) the following:

$$\delta(e_1 \ominus e_2)(d) = -\langle \omega, e_1 + e_2 \rangle \tag{2}$$

$$\delta(e_1 \ominus e_2)(l) = -2 \tag{3}$$

$$\delta(e_1 \ominus e_2)\left(\frac{d}{l}\right) = 2\frac{d}{l} - \frac{1}{l}\langle \omega, e_1 + e_2 \rangle = 0 \tag{4}$$

$$\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{l} \tag{5}$$

$$\delta^2\left(\frac{d}{l}\right) = \frac{1}{l}\delta^2(d) - \frac{2\delta(d)\delta(l)}{l^2} + \frac{2d(\delta(l))^2}{l^3} - \frac{d}{l^2}\delta^2(l) \geq 0 \tag{6}$$

2 Calculations

Now that $e_1 \neq e_2$, when we see these two vectors dotted with one another we will not be able to simplify to 1. We'll need to come up with an expression for their product:

$$\begin{aligned}
&\langle e_1 + e_2, e_1 + e_2 \rangle = |e_1 + e_2|^2 \\
&\Rightarrow 2 + 2\langle e_1, e_2 \rangle = |e_1 + e_2|^2 \\
&\Rightarrow 2\langle e_1, e_2 \rangle = |e_1 + e_2|^2 - 2
\end{aligned}$$

We'll want to keep this relationship in mind. Now, plugging in equations (2) and (3) above into equation (6), we can simplify our second variation inequality to

$$\delta^2\left(\frac{d}{l}\right) = \frac{1}{l}\delta^2(d) - \frac{4}{l^2}\langle \omega, e_1 + e_2 \rangle + 8\frac{d}{l^3} \geq 0 \tag{7}$$

Then, in light of equation (5), we can simplify the middle term as follows:

$$\delta^2\left(\frac{d}{l}\right) = \frac{1}{l}\delta^2(d) - \frac{4}{l^2}\left(2\frac{d}{l}\right) + 8\frac{d}{l^3} = \frac{1}{l}\delta^2(d) \geq 0 \tag{8}$$

Thus, we need only compute $\delta^2(d)$, which is not the same as in Case 1, as we will see when we compute it. There are two reasons for this: first of all, our variation vector is now (1, -1) as opposed to (1, 1), and secondly, we will get cross term inner products when we compute d_{uv}, which is why we needed an expression for $\langle e_1, e_2\rangle$. In Case 1 that expression was simply 1.

Now we begin our computation of $\delta^2(e_1 \ominus e_2)(d)$:

$$\begin{aligned}\delta^2(e_1 \ominus e_2)(d) &= \left\langle H(d)\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)\right\rangle = \left\langle \begin{pmatrix} d_{uu} & d_{uv} \\ d_{vu} & d_{vv}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\rangle \\ &= d_{uu} - 2d_{uv} + d_{vv} \end{aligned} \tag{9}$$

So we need to compute these second partials, starting with the relations $d_u = -\langle\omega, e_1\rangle$ and $d_v = \langle\omega, e_2\rangle$.

Starting with d_u, we get that:

$$d_{uu} = -\langle\omega_u, e_1\rangle - \langle\omega, \vec{\kappa}(u,t)\rangle \tag{10}$$

So we need to compute ω_u:

$$\omega_u = -\frac{1}{d}e_1 - \frac{X(v,t) - X(u,t)}{d^2}d_u = \frac{\langle\omega, e_1\rangle}{d}\omega - \frac{e_1}{d} \tag{11}$$

Plugging this back into equation (10) and using equation (5), we get that

$$d_{uu} = \frac{1}{d} - \frac{d}{l^2} - \langle\omega, \vec{\kappa}(u,t)\rangle \tag{12}$$

It should not be difficult, following the same procedure and using equation (5), to compute that

$$d_{vv} = \frac{1}{d} - \frac{d}{l^2} + \langle\omega, \vec{\kappa}(v,t)\rangle \tag{13}$$

Lastly, we compute that

$$d_{uv} = -\langle\omega_v, e_1\rangle = \frac{d}{l^2} - \frac{1}{d}\langle e_1, e_2\rangle \tag{14}$$

Now, we want to plug all of these into equation (9) and use our expression for $\langle e_1, e_2\rangle$. We should get that

$$\delta^2(e_1 \ominus e_2)\left(\frac{d}{l}\right) = \frac{1}{dl}|e_1 + e_2|^2 - 4\frac{d}{l^3} + \frac{1}{l}\left\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \right\rangle \geq 0 \tag{15}$$

But we will show that the first two terms are, in fact, equal. In this case $\omega \parallel (e_1 + e_2)$, so we can use the fact that $\langle \omega, e_1 + e_2 \rangle = |e_1 + e_2|$ to rewrite the first term in (15) as

$$\frac{1}{dl}\langle \omega, e_1 + e_2 \rangle^2 = \frac{1}{dl}\left(\langle \omega, e_1 \rangle + \langle \omega, e_2 \rangle\right)^2 = \frac{1}{dl}\left(\frac{d}{l} + \frac{d}{l}\right)^2 = 4\frac{d}{l^3} \tag{16}$$

Then the first two terms in (15) obviously do cancel and we wind up with the same conclusion as in Case 1, as we wanted:

$$\frac{1}{l}\left\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \right\rangle \geq 0 \tag{17}$$