High-altitude free fall

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The problem of an object falling from high altitudes where the variation of atmospheric pressure cannot be neglected is investigated. The equation of motion for the variation of the velocity of the object as a function of altitude is solved exactly. The results show that, unlike an object falling in a uniform atmosphere whose speed monotonically increases and approaches the terminal speed, the speed of a high-altitude falling object first increases, goes through a maximum, and then decreases and approaches the terminal speed from above. The results also show that if the initial altitude of the object is greater than a critical value, the object always strikes the ground with a speed that is higher than its terminal speed by a finite value, in contrast to the case of a freely falling object in a uniform atmosphere.

I. INTRODUCTION

Thirty-six years ago, on 16 August 1960, United States Air Force Captain Joseph W. Kittinger, Jr. undertook the task of carrying out a fascinating, yet very dangerous and arduous, experiment in atmospheric physics as a part of a program known as *Project Excelsior*. Kittinger's purpose was to test the apparatus necessary to allow American military pilots to safely escape by parachute from aircraft flying at altitudes above 100 000 feet (~30 500 m). To do this, Kittinger rode in an open gondola suspended below a giant plastic helium-filled balloon, which would swell to a diameter of 200 feet (~61 m) at high altitude, that would carry him from his launch point due east of the Trinity Site in southern New Mexico (location of the world's first nuclear explosion) in a circuitous route, first eastward over the Sacramento Mountains and then westward again back over the launch site in the Tularosa Basin. The balloon's ascent rate was 1200 to 1300 feet per minute (~6.1 to 6.6 m/s), and within an hour and a half of liftoff, Kittinger had reached an elevation of 102 800 feet (~31 300 m) where he was above 99% of the Earth's atmosphere. Then Kittinger walked over to the door of the gondola where a thoughtful sign reminded him that he was about to step off the "highest step in the world" and, with only a little hesitation, leaned forward and began his 16-mile (~25.7-km), four-and-a-half minute free-fall.

At first, because the atmosphere had such a low density, Kittinger felt as if he were suspended in space, not moving at all. When he rolled over and looked back up at his balloon, it seemed to shoot away from him as if it had been snapped by a huge rubber band. Of course, he knew that it was he who was moving but the absence of any "wind" confused his perception. In the early part of his fall, Kittinger was accelerating at very nearly the full acceleration due to gravity, or about 22 miles per hour each second (~9.8 m/s^2); at this altitude the acceleration due to gravity is only about 0.9% less than its value at the surface. However, after 16 s his Beaupre stabilization parachute opened, caught the slight rush of air, and he found himself descending as planned with his feet downward. He continued to accelerate at a very high rate, however, and at 90 000 feet (~27 430 m), 30 s into the fall, he reached his maximum velocity of 614 miles per hour (~274 m/s), very nearly the speed of sound at that altitude. Below that altitude the steadily increasing density of the atmosphere began to decelerate Kittinger, and as he passed through 50 000 feet (~15 240 m), he had slowed to a mere 250 miles per hour (~112 m/s). Between 50 000 and 40 000 feet (~15 200 and 12 200 m) he passed through the coldest part of the atmosphere, where the temperature dropped to nearly −100 deg F (~73 deg C below zero). He fell through 40 000 feet (~12 200 m) at 2.5 min into the jump, through 30 000 feet (~9100 m) at 3.5 min, and then through 20 000 feet (~6100 m) at 4.4 min, his speed steadily decreasing as the density of the atmosphere increased. Four minutes and 37 s into his jump, at 18 000 feet (~5500 m), his main parachute opened right on schedule and his descent rate dropped to a mere 12 miles per hour (~5.4 m/s). Then, 13 min and 45 s after leaving the balloon and with a total mass (with equipment) of 320 pounds (~145 kg), Kittinger landed without injury on the sage-covered desert west of Tularosa, NM, having completed the longest free-fall ever attempted, having demonstrated the effectiveness of the survival techniques and equipment he and his colleagues had developed, and, incidentally, having carried out a fascinating and instructive experiment in free-fall in the presence of air resistance.[1–3]

The problem of free-fall in the presence of air resistance is discussed in nearly every textbook on mechanics. The force of air resistance, $F(v)$, is not in general a simple function of velocity. However, in many cases, a good approximation can be obtained by using a combination of a linear term and a quadratic term as follows:

$$F(v)=-k_1v-k_2v|v|. \tag{1}$$

For small objects moving at low speeds, the linear term is dominant. However, even for objects of baseball size, the quadratic term dominates for speeds in excess of a few centimeters per second.[4] For speeds of the order of 24 m/s and higher but below the speed of sound, the force of air resistance is approximately given by the quadratic term only.[5] Since this speed is reached by a freely falling object in about 2.4 s, it is therefore reasonable to consider a drag force consisting of a single quadratic term, namely,

$$F(v)=-kv|v|, \tag{2}$$

for falling objects that remain in the air for times much larger than this. In this case, the equation of motion can be written as

$$mv\frac{dv}{dz}=-mg+kv^2. \tag{3}$$

For a uniform atmosphere, k is a constant and this equation can readily be integrated analytically to give

$$|v|=v_t[1-e^{-2g(z_0-z)/v_t^2}]^{1/2}, \tag{4}$$

where z_0 is the initial altitude from which the object is released from rest and z is its altitude at a later time. The terminal speed, v_t, of the object is given by

$$v_t=\left(\frac{mg}{k}\right)^{1/2}. \tag{5}$$

As Eq. (4) shows, an object released from any height hits the ground with a speed somewhat less than its terminal speed.

When an object is released from a high altitude, the assumption of a uniform atmosphere is no longer valid as the atmospheric pressure, and hence, the coefficient of drag, k, in Eq. (3) varies with altitude. Consequently, Eq. (4) will no longer represent the solution of Eq. (3). In a numerical calculation, Shea[6] noticed that if the drag force is multiplied by an exponential function of altitude, the speed of the falling object as a function of time goes through a maximum instead of attaining the classical terminal speed.

In what follows, we investigate this problem in detail by solving the equation of motion exactly. We then explain some of the interesting features of a high-altitude freely falling object resulting from the altitude dependence of atmospheric pressure.

II. THEORY

In an isothermal atmosphere, the variation of pressure as a function of altitude, z, is given by the well-known Laplace law of atmospheres[7]

$$P=P_0e^{-Mgz/RT}, \tag{6}$$

where P_0 is the pressure of the atmosphere at sea level, M is the average molar mass of the air, R is the ideal gas constant, and T is the absolute temperature. Although this equation assumes an isothermal atmosphere which, strictly speaking, is not the case, the coefficient Mg/RT can be fitted to the experimental data so that Eq. (6) can fairly well describe the variation of the atmospheric pressure with altitude. Figure 1 shows the variation of pressure as a function of altitude for the standard atmosphere.[8] To a good approximation, this variation can be described by an exponential function. An exponential least-squares fit gives a value of 1.3401×10^{-4} m^{-1} (or $0.134\,01\ \mathrm{km}^{-1}$) for the coefficient Mg/RT in Eq. (6). With an average molecular mass of 0.0288 kg/mol for the air, this gives an effective temperature of about 254 K for the atmosphere. The result is shown in Fig. 1. In the rest of this paper, we refer to the nonuniform atmosphere described by Eq. (6) as the Laplacian atmosphere, and to RT/Mg as the characteristic height of the atmosphere, λ, with a value of 7.4621×10^3 m.

The coefficient of air resistance is proportional to air density which, in turn, is proportional to air pressure in an isothermal atmosphere. Therefore, for the Laplacian atmosphere, we find

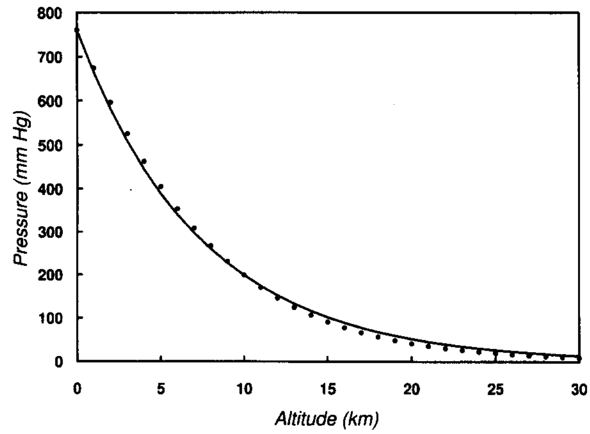

Fig. 1. Altitude dependence of atmospheric pressure. Standard Atmosphere 1962 data (dots) and their exponential least-squares fit (solid curve).

$$mv\frac{dv}{dz}=-mg+k_0v^2e^{-z/\lambda}, \tag{7}$$

instead of Eq. (3), where k_0 is the coefficient of air resistance at sea level (1 atm pressure). As can be seen, there is no terminal speed that can be obtained by setting the left-hand side of this equation equal to zero. However, we rewrite this equation as

$$v\frac{dv}{dz}=-g\left(1-\frac{k_0}{mg}v^2e^{-z/\lambda}\right), \tag{8}$$

and we continue to refer to the quantity $(mg/k_0)^{1/2}$ as the terminal speed, v_t, i.e., the terminal speed of the same object falling in an isobaric atmosphere of pressure P_0 (1 atm). Defining new variables u and x by $u\equiv k_0v^2/mg=(v/v_t)^2$ and $x\equiv\exp(-z/\lambda)$ transforms Eq. (8) into

$$\frac{du}{dx}+au=\frac{a}{x}, \tag{9}$$

where $a\equiv2g\lambda/v_t^2$ is a dimensionless constant. Multiplying both sides of this equation by the integrating factor $\exp(ax)$ gives

$$d(ue^{ax})=a\frac{e^{ax}}{x}dx. \tag{10}$$

Finally, integration of Eq. (10) gives

$$u=ae^{-ax}\int_{ax_0}^{ax}\frac{e^{\xi}}{\xi}d\xi, \tag{11}$$

where we have used the initial condition $u=0$, $x=x_0$. The exponential integral on the right-hand side of this equation is a well known function, which can also be written as a power series to give

$$u=ae^{-ax}\left[\ln\left(\frac{x}{x_0}\right)+\sum_{n=1}^{\infty}\frac{a^n(x^n-x_0^n)}{nn!}\right]. \tag{12}$$

Transforming the variables back to v and z, we get

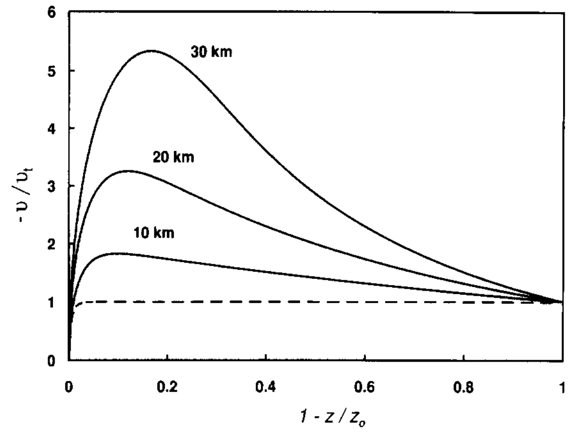

Fig. 2. Speed as a function of altitude for a freely falling object in a Laplacian atmosphere for initial altitudes of 10, 20, and 30 km. The speed of the same object falling in a uniform atmosphere (dotted curve) is also shown for comparison.

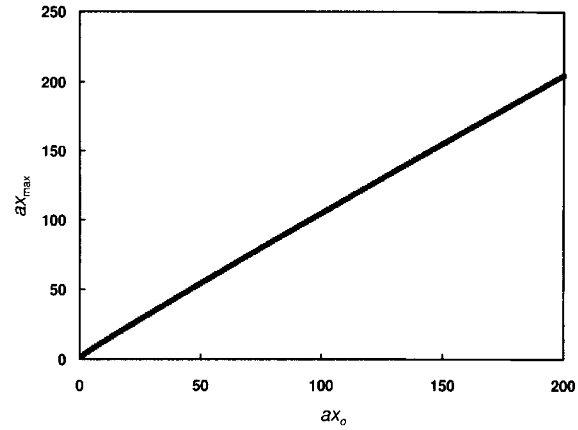

Fig. 3. Variation of the altitude at which the maximum speed occurs (z_{max}) as a function of the initial altitude (z_0). The dimensionless parameters a and x are defined by $a \equiv 2\lambda k_0/m = 2g\lambda/v_t^2$ and $x \equiv \exp(-z/\lambda)$, respectively. The dark graph is due to the congestion of 200 points that were used to construct the graph.

$$\frac{v}{v_t} = -a^{1/2}\exp[-(a/2)\exp(-z/\lambda)]\left\{\frac{(z_0-z)}{\lambda} + \sum_{n=1}^{\infty}\frac{a^n[\exp(-nz/\lambda)-\exp(-nz_0/\lambda)]}{nn!}\right\}^{1/2}, \quad (13)$$

which is an exact equation, giving the velocity of an object falling from an initial high altitude z_0 as a function of altitude. The negative sign on the right-hand side of the equation is due to the fact that the upward direction is chosen to be positive. As mentioned earlier, the quantity a is a dimensionless constant and λ is the characteristic height of the atmosphere. Again, we stress the fact that v_t is only representing the quantity $(mg/k_0)^{1/2}$. This quantity represents the terminal speed the same object would have had, if it had been falling in a uniform atmosphere. In a Laplacian atmosphere, there is no terminal speed.

III. DISCUSSION

We have checked the validity of Eq. (13) by comparing the values of the velocity obtained from this equation to those obtained from the numerical integration of the differential equation of motion, Eq. (8). We used a Euler–Richardson algorithm for an object falling freely from an initial altitude of 50 km. The numerical integration results and those obtained from Eq. (13) were identical throughout the entire motion.

The coefficient of second-order air resistance, k_0, for a spherical object is given by $k_0 = 0.22D^2$, where D is the diameter of the sphere in meters.[4] Therefore, for a pebble of mass 0.01 kg and diameter 0.02 m, we find a terminal speed of about 33 m/s. Similarly, for a skydiver of mass 70 kg and an effective diameter of about 1 m, we find a terminal speed of about 56 m/s. We use a terminal speed of $v_t = 40$ m/s throughout our calculations; however, the conclusions are general.

Graphs of Eq. (13) for three different initial altitudes are given in Fig. 2. A graph for the same object falling in a uniform atmosphere, described by Eq. (4), is also given for comparison. These graphs show several interesting features. First, unlike the case of a uniform atmosphere where the speed of the falling object quickly approaches the terminal speed, a high-altitude freely falling object increases its speed to a maximum value far above the terminal speed and then approaches the terminal speed from above. The maximum speed reached by the falling object corresponds to $dv/dt = 0$, which makes the left-hand side of Eq. (8) vanish. Therefore, the maximum speed and the altitude at which this maximum speed occurs are related by

$$\left(\frac{v_{max}}{v_t}\right)^2 = \exp\left(\frac{z_{max}}{\lambda}\right), \quad (14)$$

which is simply $u_{max} = 1/x_{max}$. Substitution into Eq. (12) gives

$$1 = ax_{max}e^{-ax_{max}}\left[\ln\left(\frac{ax_{max}}{ax_0}\right) + \sum_{n=1}^{\infty}\frac{(ax_{max})^n-(ax_0)^n}{nn!}\right], \quad (15)$$

which is a transcendental equation for the altitude at which the maximum speed occurs as a function of the initial altitude. Although this equation is fairly complicated, its solutions are very interesting. Figure 3 shows the solutions of this equation. As can be seen from this graph, to a good approximation, ax_{max} is a linear function of ax_0. In fact, a linear least-squares fit to the 200 points that construct Fig. 3 gives

$$ax_{max} = 1.011\,55 \;\; ax_0 + 3.232\,32, \quad (16)$$

with a linear correlation coefficient of 0.999 98. This linear equation reproduces the roots of Eq. (15) with an accuracy of better than 0.9% in the range of $20 \leq ax_0 \leq 200$. For $ax_0 < 20$, the accuracy decreases, however, this is the region of high altitudes. Since $x = \exp(-z/\lambda)$, we get

$$\frac{d(ax_{max})}{ax_{max}} = -\frac{dz_{max}}{\lambda}, \quad (17)$$

and since $\lambda = RT/Mg = 7.4621 \times 10^3$ m, we find an error of about 75 m in dz_{max} for every 1% error in ax_{max}. For example, for an error of 10% in ax_{max}, we get an error of about 750 m in the altitude at which the maximum speed takes place. For high altitudes, however, this error is small. Nevertheless, Eq. (16) should only be used as a quick tool for

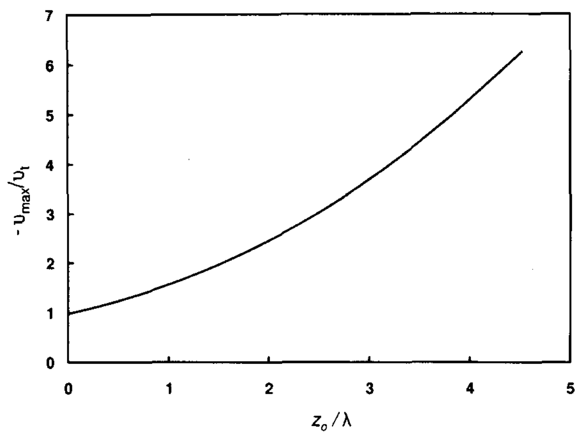

Fig. 4. Maximum speed reached by a high-altitude freely falling object as a function of its initial height in dimensionless quantities.

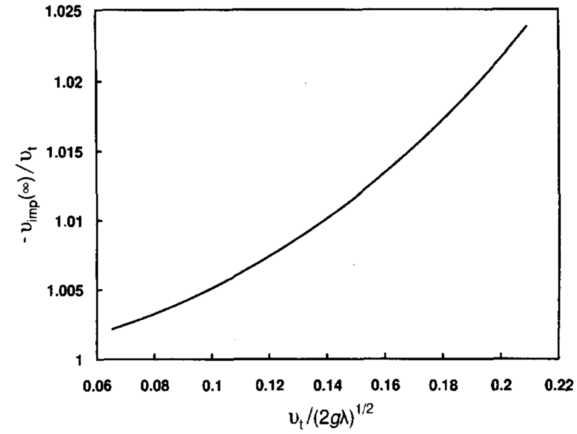

Fig. 6. Asymptotic ratio of the impact speed to the terminal speed of a high-altitude falling object in a Laplacian atmosphere as a function of its terminal speed. Note that $v_t/(2g\lambda)^{1/2}=1/a^{1/2}$ is a dimensionless quantity, with a value of 0.1045 corresponding to $v_t=40$ m/s.

estimating the altitude of maximum speed when the initial altitude is known. For exact values, Eq. (15) should be solved numerically. A remarkable feature of Eqs. (15) and (16) is that they are general relationships, independent of the values of the parameters λ and v_t, between the dimensionless quantities $ax_{\max}$ and ax_0.

Once $x_{\max}$ is obtained, the maximum speed can be calculated from Eq. (14). Figure 4 shows the maximum speed reached by a high-altitude falling object as a function of its initial height.

Another striking feature of the results is the discrepancy between the limiting value of the speed reached by a high-altitude falling object in a Laplacian atmosphere versus that reached in a uniform atmosphere. In a uniform atmosphere, a falling object very quickly approaches its terminal speed. In a Laplacian atmosphere, on the other hand, an object falling from a high altitude approaches a limiting speed that is somewhat higher that its terminal speed. This can be seen in Fig. 5, where the speed with which the object strikes the ground, the impact speed, is plotted as a function of the initial height. To do so, we have set $z=0$ in Eq. (13) and plotted v/v_t as a function of z_0/λ for an object whose terminal speed is 40 m/s. As can be seen, if the initial altitude is greater than a critical value, the object always strikes the ground with a speed that is higher than its terminal speed, and whose value increases asymptotically with the initial altitude of the object. The critical initial altitude and the asymptotic value of the impact speed, $-v_{\mathrm{imp}}(\infty)$, are both functions of the parameters used. For our choice of parameters, they are given by $z_0(\mathrm{crit})/\lambda=0.051\,082$ and $-v_{\mathrm{imp}}(\infty)/v_t=1.005\,572\,2$, respectively. Furthermore, the ratio $-v_{\mathrm{imp}}(\infty)/v_t$ increases with the terminal speed of the object, as depicted in Fig. 6.

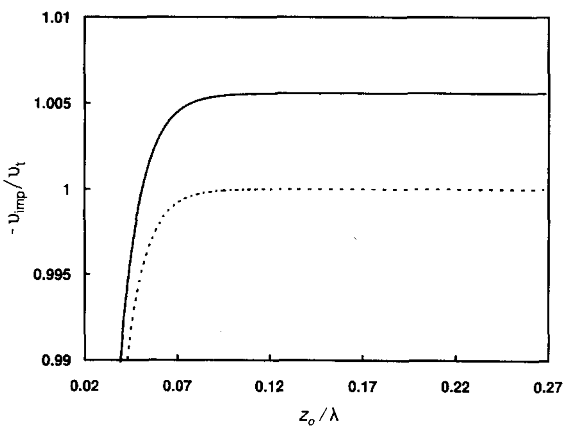

Fig. 5. Impact speed of an object falling freely in a Laplacian atmosphere as a function of its initial height in dimensionless quantities (solid curve). The graph for the same object falling in a uniform atmosphere (dotted curve) is also given for comparison. Note that if the initial altitude is greater than a critical value ($z_0/\lambda=0.051\,082$ in this case), the object always strikes the ground with a speed that is higher than its terminal speed.

Regarding Kittinger's experiment, it would be interesting to see how his data fit our model. As stated earlier, Kittinger reached a maximum speed of 274 m/s at an altitude of 27 430 m. Substituting these numbers into Eq. (14), we obtain a terminal speed of 43.7 m/s. Furthermore, since the maximum speed and the altitude at which the maximum speed takes place must also satisfy Eq. (13), we can use these values and Eq. (13) to find the value of the initial altitude, z_0, which we find to be 34 670 m. This number is different from Kittinger's actual initial altitude of about 31 300 m due to the fact that 16 s into the jump he opened his Beaupre stabilization parachute, changing the drag coefficient and, hence, the constants of the motion. With the values of z_0, and v_t thus obtained, we can now evaluate Kittinger's speed at any altitude during his fall. For example, at an altitude of 15 240 m (50 000 feet), Eq. (13) gives a speed of 129 m/s. This compares with the value of 112 m/s (250 miles per hour) reported by Kittinger. Considering the significant figures reported for the Kittinger experiment, the agreement is quite good.

IV. CONCLUSION

Due to the altitude dependence of atmospheric pressure, the dynamics of objects falling from high altitudes are quite different from those of objects falling in a uniform atmosphere. Nevertheless, the equation of motion can still be solved exactly to obtain the variation of the velocity of the object as a function of its altitude. The results show that the object's speed first increases, reaches a maximum value, and then decreases and approaches the terminal speed from above. The maximum speed reached by the falling object is a monotonically increasing function of its initial altitude. It is also found that if the initial altitude of the object is greater than a critical value, the object always strikes the ground with a speed that is higher than its terminal speed by a finite value, which increases asymptotically with the initial altitude. This is in contrast to the case of a falling object in a uniform atmosphere. The asymptotic ratio of the impact speed to the terminal speed increases with the terminal speed of the object.

Numerical solution of the equations of motion[6] for position and velocity as a function of time of an object falling through a nonuniform atmosphere is a trivial task and, therefore, we did not present them here.

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The multivariate Langevin and Fokker–Planck equations

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A novel derivation of the Langevin equation that was recently presented in this journal for a univariate continuous Markov process is generalized here to the more widely applicable multivariate case. The companion multivariate forward and backward Fokker–Planck equations are also derived. The derivations require just a few modest assumptions, and are driven by a self-consistency condition and some established theorems of random variable theory and ordinary calculus. The constructive nature of the derivations shows *why* a multivariate continuous Markov process must evolve according to equations of the canonical Langevin and Fokker–Planck forms, and also sheds new light on some uniqueness issues. The need for self-consistency in the time-evolution equations of both Markovian and non-Markovian stochastic processes is emphasized, and it is pointed out that for a great many non-Markovian processes self-consistency can be ensured most easily through the multivariate Markov theory.

I. INTRODUCTION

A recent article in this journal[1] presented a derivation of the Langevin equation for a univariate (scalar) continuous Markov process. Here we generalize that derivation to the multivariate case in which the process has $M \geqslant 1$ components, and we also derive the companion forward and backward Fokker–Planck equations. We shall presume here an acquaintance with certain parts of Ref. 1, specifically its Secs. II A–II C, so that we may avail ourselves of several important definitions and theorems introduced there; also, a familiarity with the comparatively simple derivation of the univariate Langevin equation given in the Appendix of Ref. 1 will afford a helpful perspective on our analysis here of the more complicated multivariate case.[2]

We begin with a quick review of the univariate results obtained in Ref. 1. If a function X of time t is continuous, memoryless, and stochastic—i.e., if X is a continuous Markov process—then its time evolution will be governed by an equation of the form

$$X(t+dt)=X(t)+A(X(t),t)dt + D^{1/2}(X(t),t)N(t)(dt)^{1/2}. \tag{1.1}$$

This is the (univariate) *standard form Langevin equation.* In it, dt is to be regarded as a real variable that is confined to the interval $[0,\epsilon]$, where ϵ is an arbitrarily small positive number; A and D can be any two smooth functions, with D being non-negative; and $N(t)$ is a normal random variable that has a mean 0 and variance 1, with $N(t)$ and $N(t')$ statistically independent if $t \neq t'$.

Equation (1.1) is essentially an "updating formula": Once the functions A and D have been specified, Eq. (1.1) tells us how to compute, from the value of the process at time t, its value at any infinitesimally later time $t+dt$. As was shown in Ref. 1, the functional form of this updating formula is a consequence of requiring X to be not only *continuous*, in the sense that $X(t+dt) \to X(t)$ as $dt \to 0$, and *memoryless*, in the sense that the right side does not depend on the value of X at any time before t, but also *self-consistent*: It should make no difference (statistically and to first order in dt) whether we compute the increment from t to $t+dt$ by a *single* application of the updating formula, or by *successive* applications thereof to successive subintervals of $[t,t+dt]$. The Langevin equation can be used to derive time-evolution equations for the moments of X, and it can also be used to construct numerical simulations of X.

The appearance of the factor $(dt)^{1/2}$ in the random term of