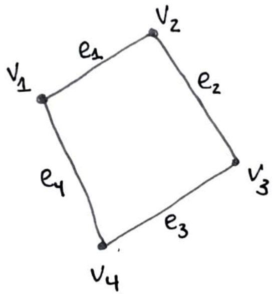
Math/CSCI 4690/6690

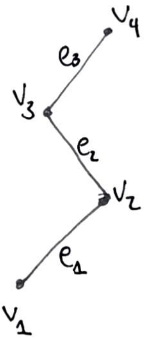
Spectral Graph Theory.

Definition. A <u>graph</u> G is a set of vertices $v_1, \dots, v_V$ and a set of edges $e_1, \dots, e_E$. Each edge joins two vertices.

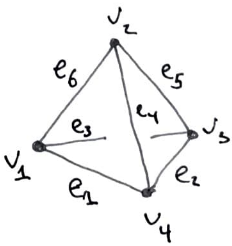
Examples.



<u>cycle graph</u> with 4 vertices



<u>path graph</u> with 4 vertices



<u>complete graph</u> with 4 vertices.

Graphs represent computer networks, friendships, chemical interactions, polymers, financial transactions, groups, Turing machines, algorithms...

In this class, we are going to learn to use matrices to analyze graphs.

Definition. The adjacency matrix M_G (often called A_G) of G is a $V \times V$ matrix defined by

$$(M_G)_{ij} = \begin{cases} 1, & \text{if an edge joins } v_i, v_j \\ 0, & \text{if not} \end{cases}$$

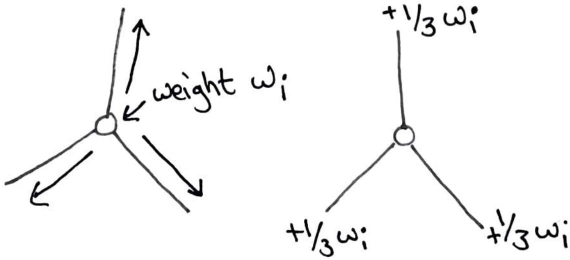
Examples.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

cycle graph — path graph — complete graph

note that M_G is always symmetric

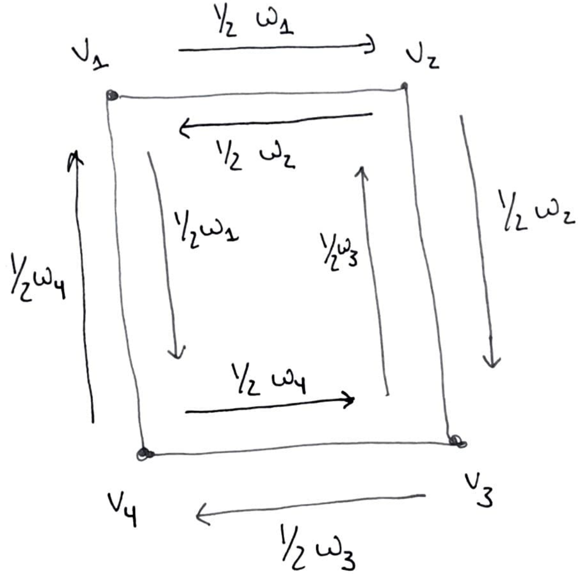
We now consider a process for graphs.



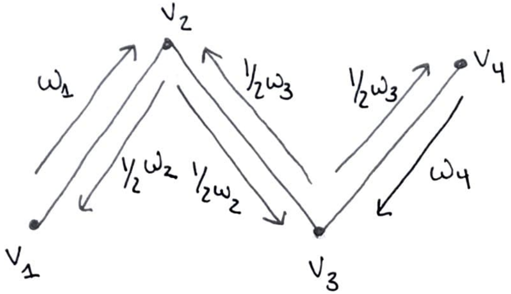
Each vertex has a weight, which it sends (equally) to each of its neighbors.

This is called the diffusion operator.

Example.

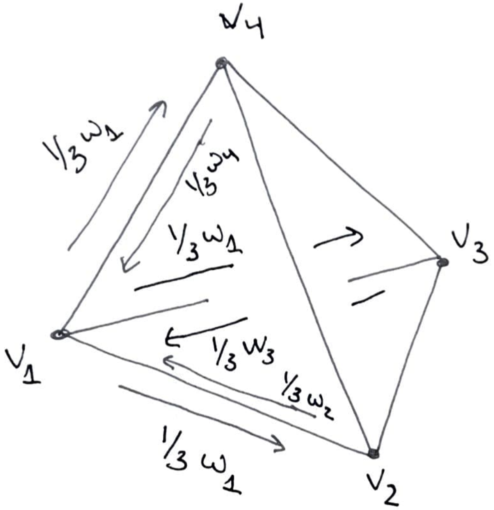


$$W_G = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\omega_2 + \frac{1}{2}\omega_4 \\ \frac{1}{2}\omega_1 + \frac{1}{2}\omega_3 \\ \frac{1}{2}\omega_2 + \frac{1}{2}\omega_4 \\ \frac{1}{2}\omega_1 + \frac{1}{2}\omega_3 \end{bmatrix}$$



$$W_G = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\omega_2 \\ \omega_1 + \frac{1}{2}\omega_3 \\ \frac{1}{2}\omega_2 + \omega_4 \\ \frac{1}{2}\omega_3 \end{bmatrix}$$

Notice! The diffusion operator is not always a symmetric matrix.



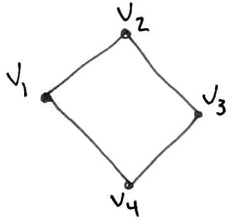
"Because symmetry" we can write

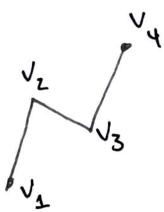
$$W_G = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} (\omega_2+\omega_3+\omega_4)/3 \\ (\omega_1+\omega_3+\omega_4)/3 \\ (\omega_1+\omega_2+\omega_4)/3 \\ (\omega_1+\omega_2+\omega_3)/3 \end{bmatrix}$$

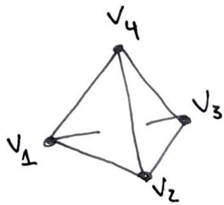
Definition. The number of edges meeting at a vertex is called the <u>degree</u> of the vertex: $\deg v_i$.

Definition. The <u>degree matrix</u> D_G is the $v \times v$ diagonal matrix whose ii-th entry is $\deg v_i$.

Examples.







$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(Recalling that the <u>inverse</u> of a square matrix M, written M^{-1} is the matrix which makes $MM^{-1} = I$.)

Proposition. The diffusion operator $W_G = M_G D_G^{-1}$.

Proof. Suppose $w_i = 1$ and all other $w_j = 0$. Then if $\delta(v_i) = (0 \cdots 0\,\underset{\uparrow \text{ ith position}}{1}\,0 \cdots 0)^T$

$$M_G D_G^{-1} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = M_G D_G^{-1} \delta(v_i)$$

$$= M_G \frac{1}{\deg v_i} \delta(v_i)$$

$$= \frac{1}{\deg v_i} \underbrace{(M_G)_{-i}}_{\text{ith column of } M_G}$$

But the ith column of M_G has 1's at the indices of neighbors of v_i and 0's everywhere else.

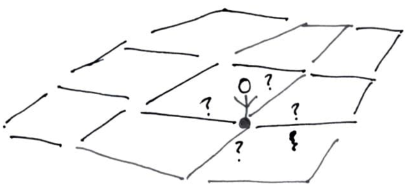
Since diffusion is linear, ~~this~~ this argument that

$$W_G \delta(v_i) = M_G D_G^{-1} \delta(v_i)$$

implies that $W_G = M_G D_G^{-1}$ as matrices.

Note: Sometimes we say that $\delta(v_i) =$ "e_i", or "the i-th standard basis vector". But we are using e_i for the ith edge, so we needed a new notation.

Another interpretation of W_G is that it describes a ~~[illegible]~~ random walk on the graph where the walker chooses from the neighbors of its current vertex with equal probability.



If ~~[illegible]~~ p is the vector where

p_i = probability that the walker is at vertex v_i, then $W_G p$ is the vector of probabilities after 1 step.

Questions.

Start at vertex v_i and walk randomly, stopping after n steps.

The probability of being at vertex v_j is given by

$$(W_G)^n \delta(v_i)$$

What is

$$\lim_{n \to \infty} (W_G)^n \delta(v_i) \, ?$$

Does it depend on v_i?

When is it the constant $\frac{1}{v} 1$?

These are some of the questions we'll answer later in the semester.