Interlude: The Ruin Problem.

We will now take a brief break to do something harder, using the ideas we've developed so far.

Q: Suppose two players engage in a betting game where player I starts with x dollars and player II starts with N-x dollars.

At each round, they bet 1 dollar, and player I wins with probability p. The game ends when one player or the other runs out of money.

What is the probability

$$P(x) = P(\text{player I wins} \mid x)?$$

We are going to see this problem as a graph

where node K represents the state "player I has K dollars".

We define

p_K = probability of moving to ~~state~~ node $K+1$ from node K.

Suppose there exist numbers C_K so that

$$p_K = \frac{C_{K+1}}{C_K + C_{K+1}}$$

and let $r_K = 1/C_K$.

Example. If all $p_k = 1/2$,

all $C_k = 1$ is an ok solution, as

$$p_k = \frac{C_{k+1}}{C_k + C_{k+1}} = \frac{1}{1+1} = 1/2$$

For other p_k, ~~we'll show later how to~~ we observe that if $q_k = 1 - p_k$, then

$$q_k = 1 - \frac{C_{k+1}}{C_k + C_{k+1}}$$

$$= \frac{C_{k+1} + C_k - C_{k+1}}{C_k + C_{k+1}}$$

$$= \frac{C_k}{C_k + C_{k+1}}$$

So $$\frac{p_k}{q_k} = \frac{C_{k+1}}{C_k}.$$

This means that

$$C_2 = C_1 \frac{p_1}{q_1}$$

$$C_3 = C_1 \frac{p_1 p_2}{q_1 q_2}$$

$$\vdots$$

$$C_{N-1} = C_1 \frac{p_1 \cdots p_{N-2}}{q_1 \cdots q_{N-2}} \qquad \square.$$

Now we're going to do something cool! Suppose our graph is composed of resistors with resistance r_k and conductance c_k, and we add a battery so that the voltage $v(x)$ ~~is~~ has

$$v(0) = 0, \qquad v(N) = 1.$$

We will show that $p(x) = v(x)$ everywhere, and use this to compute $p(x)$ explicitly!

Step 1. $p(0) = 0$, $p(N) = 1$.

This is obvious. At 0, player I has no money left to bet, so they cannot win. At N, player I has all the money, so they cannot lose.

To prove this in the middle, we need some new ideas.

Definition. Given C_K associated to the edges ~~to~~ of a linear graph, we say $f(x)$ is <u>harmonic</u> if

$$f(x) = \frac{C_{x+1}}{C_x + C_{x+1}} f(x+1) + \frac{C_x}{C_x + C_{x+1}} f(x-1)$$

That is, $f(x)$ is a weighted average of $f(x+1)$ and $f(x-1)$.

Step 2. $p(x)$ is harmonic.

We know that if player I has x dollars at some point, in the next round, pI will have either ~~x~~ x+1 or x−1 dollars.

$p(x) = P(\text{pI wins} \mid \text{pI has x dollars})$

$= p_k\, p(x+1) + q_k\, p(x-1)$

$= P(\text{pI wins next game})$ ~~P(pI wins~~ $\times$
$P(\text{pI wins} \mid \text{pI has x dollars and wins next game})$

$+\, P(\text{pI loses next game}) \times$
$P(\text{pI wins} \mid \text{pI has x dollars and loses next game})$

$$= \frac{C_{k+1}}{C_k + C_{k+1}}\, p(x+1) + \frac{C_k}{C_k + C_{k+1}}\, p(x-1).$$

Step 3. $v(x)$ is harmonic.

Kirchhoff's current law says that

the current i flowing into and out of node x is the same.

Ohm's law says

$$i = \frac{v(x) - v(x-1)}{r_x}$$

$$= \frac{v(x+1) - v(x)}{r_{x+1}}$$

Recalling that $r_x = \frac{1}{c_x}$,

$$C_x v(x) - C_x v(x-1) = C_{x+1} v(x+1) - C_{x+1} v(x)$$

$$((C_x + C_{x+1}) v(x) = C_{x+1} v(x+1) + C_x v(x-1)$$

$$v(x) = \frac{C_{x+1}}{C_x + C_{x+1}} v(x+1) + \frac{C_x}{C_x + C_{x+1}} v(x-1). \qquad \square$$

Step 4. The maximum and minimum of a harmonic function are attained at the boundary.

Proof. Suppose $f(x) = M$. Then we claim $f(x-1) = M = f(x+1)$, for other wise

$$M = f(x) = \frac{C_{x+1}}{C_x + C_{x+1}} f(x+1) + \frac{C_x}{C_x + C_{x+1}} f(x-1)$$

$$< \frac{C_{x+1}}{C_x + C_{x+1}} M + \frac{C_x}{C_x + C_{x+1}} M = M. \quad ⨳$$

Step 5. If f, g are harmonic, then $f+g$ and kf are harmonic.

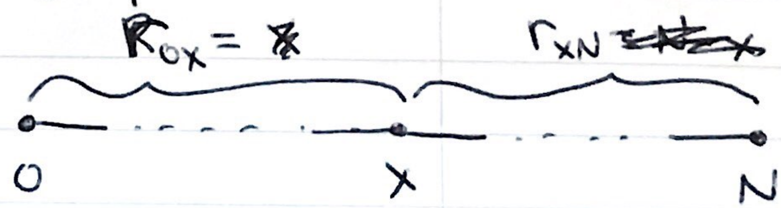
Proof. Just regroup the definitions.

Step 6. If f, g are harmonic and $f(0)=g(0)$, $f(N)=g(N)$ then $f(x)=g(x)$.

By Step 5, $f-g \overset{h}{=}$ is a harmonic function with $h(0)=h(N)=0$. By Step 4, this means the max and min of h are 0, so $h(x)=0$ everywhere.

~~Step 7.~~

We now know $p(x)=v(x)$. We can use this to finish the problem.



We Know

$$\frac{v(x)-v(0)}{r_{0x}} = i = \frac{v(N)-v(x)}{r_{Nx}}$$

and $r_{0x} = r_1 + \ldots + r_{x-1}$ while $r_{Nx} = r_{x+1} + \ldots + r_{N-1}$. So, using $v(0)=0$, $v(N)=1$, we get

$$\frac{v(x)}{\sum_{i=1}^{x-1} r_i} = \frac{1-v(x)}{\sum_{i=x+1}^{N-1} r_i}$$

$$\left(\sum_{i=x+1}^{N-1} r_i\right) v(x) = \sum_{i=1}^{x-1} r_i - \left(\sum_{i=1}^{x-1} r_i\right) v(x)$$

or

$$v(x) = \frac{\sum_{i=1}^{x-1} r_i}{\sum_{i=1}^{N} r_i}$$

Now if $p = 1/2$, $r_i = c_i = 1$, so we get

$$v(x) = p(x) = \frac{x}{N}.$$

In general, $r_i = \frac{1}{c_i}$, and so

$$r_i = \frac{1}{c_1} \frac{q_1 \cdots q_{i-1}}{p_1 \cdots p_{i-1}}$$

so

$$p(x) = \frac{\frac{1}{c_1} + \frac{1}{c_1}\frac{q_1}{p_1} + \cdots + \frac{1}{c_1}\frac{q_1 \cdots q_{x-1}}{p_1 \cdots p_{x-1}}}{\frac{1}{c_1} + \frac{1}{c_1}\frac{p_1}{q_1} + \cdots + \frac{1}{c_1}\frac{q_1 \cdots q_{N-1}}{p_1 \cdots p_{N-1}}}$$

Exercise: Simplify this formula when $p_1 = \cdots = p_N = p$, $q_1 = \cdots = q_N = q = 1 - p$.