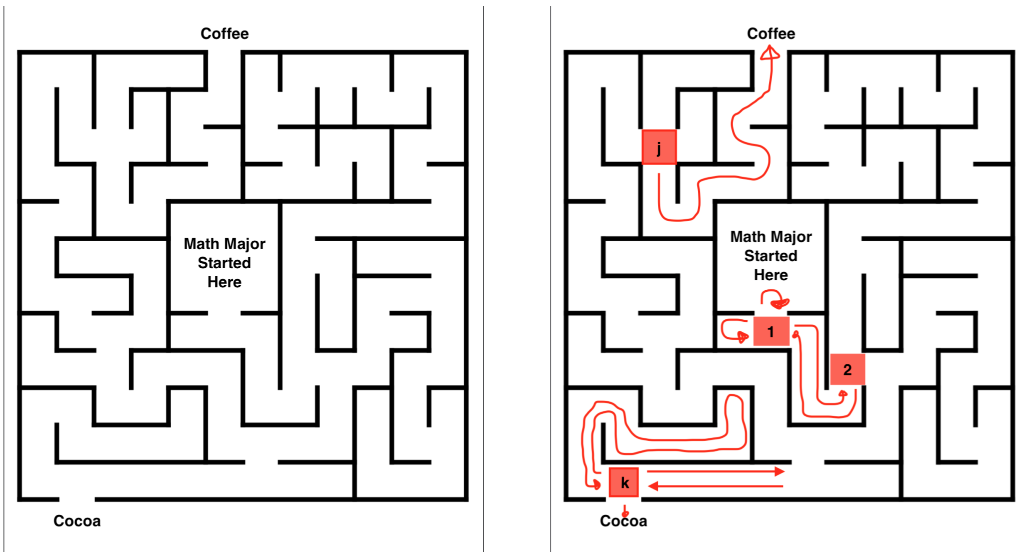
Math 4600/6600

1. Problems about Markov Chains

(1) **Are you smarter than a Roomba?** A psychology lab conducted an experiment to see whether math majors prefer the smell of coffee or cocoa. In the experiment, majors were placed in the central room of the Psychology building, which has two exits. At one exit, the researchers provided coffee, while at the other exit, the researchers provided cocoa. Majors who left the building by either door were not permitted to return, and were recorded as preferring the corresponding beverage.

Here is a map of the psychology building:



In the experiment, 75% of the math majors left by the door providing cocoa. Does this mean that math majors prefer cocoa? Does it mean anything?

We're going to think about these questions by comparing the performance of the average math major with a simple model for how a Roomba navigates a maze. To build our model, we first find every square in the maze with more than one missing wall and call it a "junction" square. These squares will be the transient states of our chain. All other squares will be "hallway" squares.

Every time the Roomba enters a junction square, it chooses a direction randomly. It proceeds in the same direction down the corresponding hallway, turning back if it reaches a dead end, until it reaches a junction square again. (This could be the same junction square if it went down a blind hallway and turned back.) The junction squares labeled j and k are adjacent to the two absorbing states ("coffee" and "cocoa").

(a) Label the maze picture with the states of your Markov chain, and write down the transition matrix

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}.$$

(b) In the experiment, 75% of the math majors left by the "cocoa" door. What fraction of Roombas leave by that door? (Do Roombas even drink coffee?)

(c) The average math major passed through 34.3 junction squares on their way through the maze. What is the expected number of junction squares passed through by the average Roomba? Are math majors better or worse than Roombas at solving mazes?

(d) Math majors often solve mazes by the "always turn left" strategy. How many junction squares will a major pass through using this strategy? Which door will they leave by?

(2) **The streak chain** Suppose that a certain experiment has K possible outcomes, each equally likely and that $x_1x_2\cdots x_n$ is a list of particular outcomes we're looking for. What is the expected number of trials before we see $x_1\cdots x_n$ (consecutively and in-order)?

We can analyze this situation using the idea of *streaks*. If the streak score is k, then the outcomes of the last k trials were $x_1\cdots x_k$. Depending on the outcome of the next trial, we either add 1 to the streak score (if the outcome is x_{k+1}) or reduce the streak score *to the largest k so that the last k trials are $x_1\cdots x_k$*. If the streak score ever reaches n we have matched the entire string and we stop.

This is an absorbing Markov chain where the states are labeled by the streak score.

Suppose that we are given a random string of digits (0-9) where each digit occurs with equal probability.

(a) Write down the streak chain for the string 7777777. Find the expected number of digits before we see this string.
(b) Write down the streak chain for the string 1212456. Be very careful at the bold digit 4. Find the expected number of digits before we see this string.
(c) Write down the streak chain for your birthday (expressed in MMDDYY format). Find the expected number of digits before we see this string.
(d) Visit the pi-search page (Google it) and search for the first occurrence of all three strings in the digits of π. Where do these strings occur in the digits of π? Do the digits of π act like random numbers?

(3) **Harmonic functions and fair games (Graduate)** Consider an absorbing Markov chain with state space S. Let f be a function defined on S with the property that

$$f(i) = \sum_{j \in S} p_i j f(j), \quad \text{or} \quad f = Pf.$$

Then f is called a harmonic function for P. If you imagine $f(i)$ to be the "amount of money" or *fortune* you have in state i, then the harmonic condition means the game is *fair* in the sense that your expected fortune after 1 step is the same as it was before the step.

(a) Show that for f harmonic, $f = P^n f$ for all n.
(b) In particular, show that if

$$P^\infty = \lim_{n\to\infty} P^n = \begin{pmatrix} 0 & (I-Q)^{-1}R \\ 0 & I \end{pmatrix}$$

then $f = P^\infty f$.
(c) Now prove that if you start in transient state i your expected final fortune is $\sum_k ((I-Q)^{-1}R)_{ik} f(k)$ and that this is equal to $f(i)$. That is, a fair game is fair to the end.

(4) **Tak (Graduate students)** Consider a board game called *tak*, which is played on an $n \times n$ board where each square is initially green or red. At each stage, a square A is chosen at random, and one of its eight neighbors B is chosen at random. The color of square B is then changed to that of square A. (We assume that the edges of the board "wrap around" so that every square *has* eight neighbors.) The two absorbing states for this Markov chain are the configurations where the board is all red or all green, in which case red (or green) wins.

Assume that your fortune at any time is equal to the proportion of red squares on the board. Give an argument that this is a *fair* game in the sense of the previous exercise. (Hint: Show that for every possible transition which increases your fortune by one, there is a corresponding one which decreases your fortune by one.) Now use the result of the last exercise to show that the probability that red will win is equal to the proportion of red squares in the initial configuration of the board.