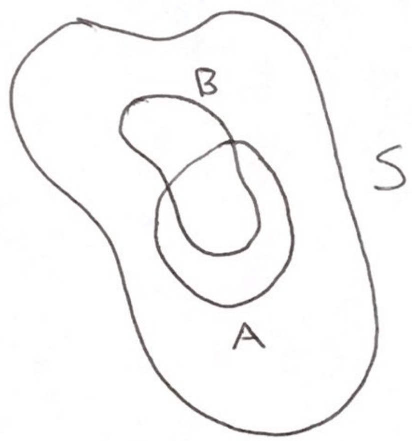
# Math 4600 – Conditional Probability

Definition. The conditional probability of ~~A~~ event A given event B is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example.




P(A) is quite small, as A is a small part of S.

But $P(A|B)$ is much larger as $A \cap B$ is a large part of B.

Remark. A and B are independent $\iff P(A)P(B) = P(A \cap B)$. In this case,

$$P(A) = \frac{P(A \cap B)}{P(B)} = P(A|B)$$

$$P(B) = \frac{P(A \cap B)}{P(A)} = P(B|A)$$

Theorem. If $P(B) > 0$, A and B are independent $\iff P(A) = P(A|B)$.

That is, B happening does not affect the probability of A.

Example. Suppose we roll a fair 6-sided die.

$A = \{1, 2, 3\}$

$B = \{3, 4\}$

$C = \{1, 2, 3, 5, 6\}$

Compute $P(A)$, $P(B)$, $P(C)$ and

$P(A|B)$, $P(B|A)$, $P(A|C)$, $P(C|A)$

$P(B|C)$, $P(C|B)$.

We notice that $P(C|B)$ and $P(B|C)$ are different. This is usually the case.

Proposition. $P(A|B) = P(B|A) \iff P(B) = P(A)$.

Proof.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ so } P(B)\,P(A|B) = P(A \cap B)$$

Similarly, $P(A \cap B) = P(A)\,P(B|A)$. Thus

$$P(A)\,P(B|A) = P(B)\,P(A|B). \quad \square$$

Example. In 2014, 69% of US teens (10-19) owned an iPhone*. 50% of U.S. teens used Snapchat*. Suppose that 60% of iPhone-owning teens use Snapchat. You get a Snap from a teenager. What is the probability that it comes from an iPhone?

$S$ = all US teens $P(S) = 1$

$A$ = uses iPhone $P(A) = 0.69$

$B$ = uses Snap $P(B) = 0.5$

$P(B|A)$ = uses Snap, given that uses iPhone $= 0.6$

We want to compute

$P(A|B)$ = uses iPhone, given that uses Snap.

We need to know $P(A \cap B)$. Now

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

so

$$P(A \cap B) = P(A)\,P(B|A) = 0.69 \times 0.6$$
$$= 0.414$$

So we compute

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0.414}{0.5} = 0.828$$

We now want to prove that conditional probabilities are probabilities - they satisfy the three axioms. We recall

Theorem. (Distributive laws)

$$\left(\bigcup_j A_j\right) \cap B = \bigcup_j (A_j \cap B)$$

$$\left(\bigcap_j A_j\right) \cup B = \bigcap_j (A_j \cap B)$$

These are easier to see in terms of <u>and</u> and <u>or</u>.

$\left(\bigcup_j A_j\right) \cap B$ = (at least one $A_j$) and $B$

$\bigcup_j (A_j \cap B)$ = at least one ($A_j$ and $B$)

$\left(\bigcap_j A_j\right) \cup B$ = (all $A_j$) or $B$

$\bigcap_j (A_j \cup B)$ = all ($A_j$ or $B$) $\square$

We now prove

Theorem. Consider an event $B$ with $P(B) > 0$.

1. For any ~~A~~ event $A$, $0 \leq P(A|B) \leq 1$.
2. For the sample space $S$, $P(S|B) = 1$.
3. If events $A_j$ are disjoint,

$$P(\cup A_j \mid B) = \sum_j P(A_j \mid B).$$

Proof.

1. Since $A \cap B \subset B$, $P(A \cap B) \leq P(B)$.
$P(A \cap B) \geq 0$ by Axiom 1. So

$$0 \leq P(A \cap B) \leq P(B)$$

and (dividing by $P(B)$)

$$0 \leq \frac{P(A \cap B)}{P(B)} \leq 1.$$

2. Since $S \cap B = B$, $P(S \cap B) = P(B)$
and

$$P(S|B) = \frac{P(S \cap B)}{P(B)} = 1.$$

3. (This is more fun).

$$P\left(\bigcup_j A_j \mid B\right) = \frac{P\left(\left(\bigcup_j A_j\right) \cap B\right)}{P(B)}$$

distributing

$A_j$ are disjoint so $A_j \cap B$ are disjoint

$$= \frac{P\left(\bigcup_j (A_j \cap B)\right)}{P(B)} = \frac{\sum_j P(A_j \cap B)}{P(B)}$$

$$= \sum_j P(A_j \mid B).$$

Example. (Simple Roulette).

A spinner has numbers

$$\underbrace{1, 2, 3, 4, 5}_{\text{red}}, \underbrace{6, 7, 8, 9}_{\text{white}}$$

all numbers are equally likely.

A = # is odd

B = # is white.

Find P(A), P(B), P(A|B), P(B|A) and describe each in words.

Example. (Books)

Randomly open a 300 page book.

1. Given that at least 1 digit of the page number is 5, find the probability that the page number is 255.

2. Given that at least 2 digits of the page number are 5, find the probability that the page number is 255.

Example 2. (Double double dice)

Roll two dice and add the results.

If the number is 7, you lose.

Otherwise, call this number $x_1$

Roll again until you get a 7 (you lose) or you get $x_1$ (you win).

What are your odds of winning?

Note. If $B_j$ form a partition of $S$,

$$P(A) = P\left(\bigcup_j (A \cap B_j)\right) = \sum_j P(A \cap B_j)$$

since $\bigcup_j (A \cap B_j) = A$, and the $A \cap B_j$ are disjoint.