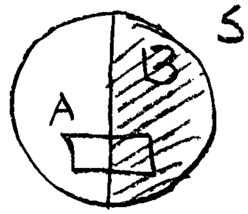
4600 - Probability - Lecture 3

__Definition.__ Events A and B are __independent__ if $P(A \cap B) = P(A)\,P(B)$, and __dependent__ otherwise.

Example.

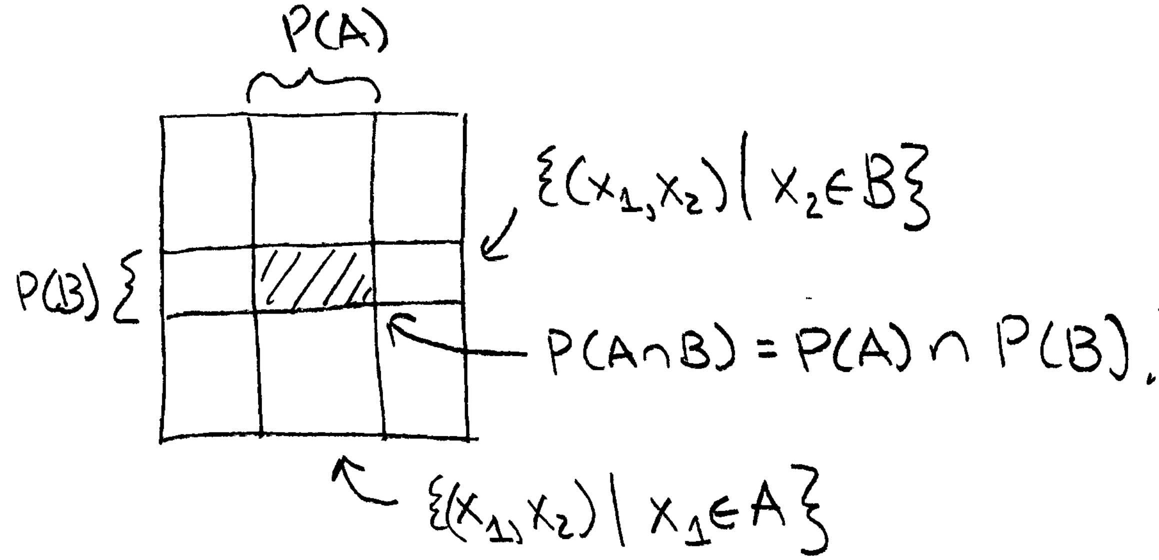


$$P(A \cap B) = \tfrac{1}{2} P(A) = P(A)P(B)$$

This is easier to draw if S has a special form. Given any probability [illegible] S with n equally likely outcomes, we can ~~describe~~ the probability of

__Definition.__ When a collection of random events are observed, we call each one a __trial__. or a __draw__.

If we take 2 trials __from the same space S__, the pair (x_1, x_2) is selected from $S \times S$.



In this case, any event in the form $\{(x_1, x_2) \in S \times S \mid x_1 \in A\}$ and any event in the form $\{(x_1, x_2) \in S \times S \mid x_2 \in B\}$ are <u>independent</u>.

Example. We flip a (fair) coin twice. Here are three events:

$$A = \{(x_1, x_2) \mid x_1 = H\}$$

$$B = \{(x_1, x_2) \mid x_2 = H\}$$

$$C = \{(x_1, x_2) \mid x_1 = x_2 = H\}$$

$$D = \{(x_1, x_2) \mid x_1 = x_2 = T\}$$

Which pairs of A, B, C, D are independent?

Example. We roll a fair (6-sided) die.

$A = \{1, 2, 3\}$

$B = \{3, 4\}$

$C = \{1, 2, 3, 5, 6\}$

Which pairs are independent?

Lemma. A and B are independent and disjoint $\Leftrightarrow$ $P(A)$ or $P(B) = 0$ (or both)

Theorem. If $A \subset B$ and $P(A) \neq 0$ and $P(B) \neq 1$, then A, B are dependent.

Proof. $P(A \cap B) = P(A)$ so $P(A \cap B)\, P(B) = P(A) P(B)$

Theorem. If neither $P(A)=0$ nor $P(A)=1$ then A, A^c are dependent.

We can extend the definition of independence to multiple events.

Definition. A collection of events A_j is (mutually) independent if for every subcollection C of the A_j,

$$P\left(\bigcap_{j\in C} A_j\right) = \prod_{j\in C} P(A_j)$$

Note. This is true for any collection of trials ~~as long as~~ $(x_1, \dots, x_j, \dots)$ as long as event A_j only depends on x_j.

Example. We flip a coin repeatedly. The individual flips are independent. (Note. The coin doesn't have to be fair)

Example. Kroger sells bags of 20 onions which contain 1 bad onion in each. bag.

A = an onion picked from bag A is bad

B = an onion picked from bag B is bad

C = a second onion picked from bag A is bad.

Which of these events are independent?

Example. We flip a coin until a total of 10 heads appear. $A_k = \{\#\text{flips between } k\text{th head and } (k+1)\text{st head} > 2\}$.

Are the A_k independent? (Discuss).

Example. A student rolls 7 (fair) dice, yielding numbers $(x_1, \ldots, x_7)$, each between 1 and 6.

$A = x_7$ is even (# on die 7 is even)

$B = x_{(x_7)}$ is even (# on die (whatever # was rolled on die 7) is even)

Are A and B independent? (Discuss.)

Example. A student rolls 3 dice which are unfair ($P(1) = 1/3$, $P(2) = 0$, $P(3) = P(4) = P(5) = P(6) = 1/6$) and 4 dice which are fair, producing 7 numbers $(x_1, \ldots, x_7)$. The ~~7th die is known to be fair.~~ dice are ordered (unfair, unfair, unfair, fair, fair, fair, fair).

Are the two events above dependent or independent? (Discuss).