Chapter 9 - Independence & Conditioning

Definition. Given a pair of discrete random variables X, Y, the joint pmf is

$$P_{X,Y}(x,y) = P(X = x \text{ and } Y = y)$$

The joint CDF is

$$F_{X,Y}(x,y) = P(X \leq x \text{ and } Y \leq y)$$

Example. Roll 2 dice, let X be the minimum and Y be the maximum of the two values.

$P_{X,Y}(3,5) =$

$P_{X,Y}(5,3) =$

$P_{X,Y}(4,4) =$

Example. Flip a coin 3 times and let X = # of heads, Y = # of tails.

$$P_{X,Y}(3,0) = 1/8 = \{H,H,H\}$$
$$P_{X,Y}(2,1) = 3/8 = \{T,H,H\},\{H,T,H\},\{H,H,T\}$$
$$P_{X,Y}(1,2) = 3/8$$
$$P_{X,Y}(0,3) = 1/8$$

If we are given a <u>joint</u> distribution of two variables, $P_{X,Y}$, we ~~makey~~ ~~ex~~ extract the <u>marginal</u> distributions ~~of one by~~

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

Example. If we roll two dice and X = min, Y = max

$$P_X(1) = \sum_{y=1}^{6} P_{X,Y}(1,y)$$

$$= 1/36 + 2/36 + 2/36 + 2/36 + 2/36 + 2/36$$

$= 11/36$

$$P_X(2) = \sum_{y=1}^{6} P_{X,Y}(2, y)$$

$= 0 + 1/36 + 2/36 + 2/36 + 2/36 + 2/36$

$= 9/36$

$P_X(3) = 0 + 0 + 1/36 + 2/36 + 2/36 + 2/36$

$= 7/36$

$P_X(4) = 0 + 0 + 0 + 1/36 + 2/36 + 2/36$

$= 5/36$

$P_X(5) = 3/36$

$P_X(6) = 1/36$

Note that $11 + 9 + 7 + 5 + 3 + 1 = 36$, so all the probability mass is accounted for.

Definition. We say that X, Y are <u>independent</u> random variables iff $p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$.

Example. ~~Let X~~ As above, roll 2 dice and let $X = \min$, $Y = \max$.

We know $p_X(1) = 11/36$. It is not hard to see $p_Y(1) = 1/36$.

$$p_{X,Y}(1,1) = 1/36$$

$$p_{X}(1)\, p_Y(1) = 11/36 \cdot 1/36$$

So these r.v.s are <u>not</u> independent.

Example. Roll 1 die and let $X = 1$ if the result is $1, 3, 5$ (odd) and $X = 0$ if the result is even. Let $Y = 1$ if the result is 5 or 6 and 0 otherwise.

Compute $p_{X,Y}$ for $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Also compute p_X for $0, 1$ and p_Y for $0, 1$.

This is an example of a more general phenomenon.

Definition. X is an indicator for event A if $X=1$ when A occurs and $X=0$ when A^c occurs.

Theorem. Let X, Y be indicators for A, B. X, Y are independent r.v.s $\Leftrightarrow$ A, B are independent events.

Example. Flip a coin until you get heads. Let $X=1$ if an even number of flips are needed and $X=0$ otherwise. Let $Y=1$ if 11 or more flips are needed and $Y=0$ otherwise.

Decide if independent.