Partial Differential Equations 2.

Consider now the case of a second order linear differential operator. If we let

$$L = \sum a_{ij}(\vec{x}) \frac{\partial^2}{\partial x_i \partial x_j} + \text{lower order stuff}$$

then the second order part is described by the quadratic form

$$\chi_L(\vec{x}, \vec{v}) = \langle A(\vec{x}) \vec{v}, \vec{v} \rangle$$

where $A(\vec{x}) = [a_{ij}(\vec{x})]$ may as well be a symmetric matrix. Then we have various cases:

1) χ_L is positive (or negative) definite $\iff$ L is elliptic

2) X_L is indefinite and has (say) only one negative eigenvalue but all other eigenvalues positive $\iff$ L is (properly) hyperbolic (cf. Courant & Hilbert p 182, vol 2)

3) X_L is singular (and so has one or more zero eigenvalues) $\iff$ L is parabolic

4) X_L has ~~some~~ >1 negative <u>and</u> >1 positive eigenvalues, which means X_L is indefinite. ~~Then~~ $\iff$ L is "ultrahyperbolic".

This makes most sense when X_L is a 2×2 matrix, in which case everything is elliptic, parabolic, or hyperbolic.

Examples.

1) Laplacian. $L = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, $\chi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Elliptic.

2) Heat operator. $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$.

$$\chi_L = \begin{bmatrix} 0 & 0 & & \\ 0 & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}.$$ Parabolic.

3) Wave operator. $L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$

$$\chi_L = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}.$$ Hyperbolic.

These give rise to the standard equations of mathematical physics.

The Laplace Equation.

$$\Delta u = 0.$$

The Poisson Equation.

$$\Delta u = f.$$

The Heat Equation.

$$\frac{\partial u}{\partial t} = \Delta_x u$$ (Δ_x means "Laplacian in the space variables)

or

$$Hu = 0$$ ($H = \frac{\partial}{\partial t} - \Delta_x$ is the heat operator)

The Wave Equation.

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u \text{ or } Wu = 0.$$

For each of these problems, we can specify boundary values (or normal derivatives), on $\partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is the domain of problem.

The basic facts are these:

- Solutions to elliptic boundary ~~values~~ value problems are as smooth (on int Ω) as the coefficients in the operator, even if the boundary data is not smooth.

- Solutions to parabolic boundary value problems are (likewise) smooth for $t>0$, (where t is the "special" variable by convention), even if the $t=0$ data is not smooth.

- Solutions to hyperbolic bvps are only ever as smooth as the boundary data.

Example. Consider Laplace's equation

$$\Delta u = 0 \text{ on } [0,1] \times [0,1].$$

with boundary conditions.

We want to solve this equation. Notice that if $u_n(x,y) = \sinh(n\pi y)\sin(n\pi x)$ then (for any n)

$$\Delta u_n = \frac{\partial^2}{\partial x^2} u_n + \frac{\partial^2}{\partial y^2} u_n$$

$$= (n\pi)^2 u_n - (n\pi)^2 u_n$$

$$= 0$$

Of course, on the boundary we have

$$u_n(0,y) = u_n(\pi,y) = u_n(x,0) = 0.$$

On $y=1$, we have

$$u_n(x,1) = \sinh(n\pi)\sin(n\pi x).$$

$$= K_n \sin(\pi n x).$$

where K_n is some constant. Now ~~u(x,1)~~ $f(x)$ has a Fourier expansion on $x \in [0,1]$ so

$$\cancel{u(x,1)}\ f(x) = \sum c_n \sin(\pi n x)$$

To match $f(x) = u(x,1)$ we need only take a linear combination of u_n to match coefficients

$$u = \sum \frac{c_n}{K_n} u_n(x,y).$$