Systems of ODE - Numerical Issues

We now introduce (or recall) a neat trick from the theory of ODE.

Suppose we have an ODE

$$\varphi''(t) = (1-\varphi) + \frac{4}{\varphi^3}$$

and we want to solve. The standard trick is to introduce dummy variables

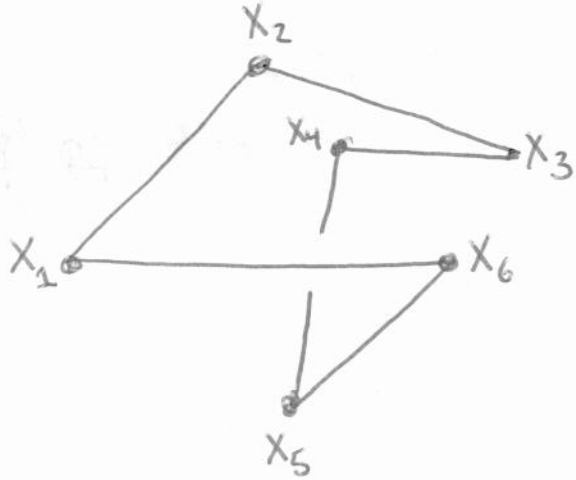
$$x(t) = \varphi(t)$$
$$y(t) = \varphi'(t)$$

and write this as a <u>system</u>

$$x'(t) = y(t)$$
$$y'(t) = 1 - x(t) + \frac{4}{x(t)^3}$$

of first order ODEs.

Example. Often we just have a high-dimensional system.



This is a point in

$$\mathrm{Pol}_6(\mathbb{R}^3) = \text{6 vertex (equilateral) polygons in } \mathbb{R}^3$$

We can describe an ODE on this space by

$$x_i'(t) = \left(x_{i-1}(t) - x_i(t)\right) \times \left(x_i(t) - x_{i-1}(t)\right)$$

This is a nonlinear system of 18 equations (one for each coordinate of the vertices).

We write such a system as

$$\vec{X}'(t) = \vec{F}(t, \vec{X}(t))$$

Now we know that the vector functions have Taylor expansions in the variables $t, \vec{X}$. As before

$$\vec{X}(t+h) = \vec{X}(t) + h\vec{X}'(t) + \frac{h^2}{2}\vec{X}''(t) + \dots$$

The expansion for $\vec{F}$ (in t) is ~~simaf~~ similarly ~~b~~ interesting

$$\vec{F}(t+h, \vec{x}+\vec{K}) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial x^\alpha} \overrightarrow{(h,K)}^{\alpha}$$

where (if $\vec{x} \in \mathbb{R}^n$) the multi-index α is a tuple

$$\alpha = (\alpha_1, \dots, \alpha_{n+1})$$

obeying

$$|\alpha| = \alpha_1 + \dots + \alpha_{n+1}$$

$$\alpha! = \alpha_1! \cdots \alpha_{n+1}!$$

$$\vec{X}^{\alpha} = X_1^{\alpha_1} \cdots X_{n+1}^{\alpha_{n+1}}$$

and we think of $(h, \vec{K})$ as a vector in $\mathbb{R}^{n+1}$.

Using these, we can rederive the general formulae for the Taylor and RK methods and they come out just the same! Specifically, RK4 is

$$\vec{X}(t+h) = \vec{X} + \frac{h}{6}(\vec{K}_1 + 2\vec{K}_2 + 2\vec{K}_3 + \vec{K}_4)$$

where

$$\vec{K}_1 = \vec{F}(t, \vec{X})$$
$$\vec{K}_2 = \vec{F}(t + \tfrac{1}{2}h, \vec{X} + \tfrac{1}{2}h\vec{K}_1)$$
$$\vec{K}_3 = \vec{F}(t + \tfrac{1}{2}h, \vec{X} + \tfrac{1}{2}h\vec{K}_2)$$
$$\vec{K}_4 = \vec{F}(t+h, \vec{X} + h\vec{K}_3)$$