We finish by proving

Lemma. If $\|X\| < 1$, then $I - X$ is invertible, $(I-X)^{-1} = \sum_{i=0}^{\infty} X^i$; $\|(I-X)^{-1}\| \leq \frac{1}{1-\|X\|}$.

Numerical Linear Algebra 2.

Last time, we saw if

$$Ax = b \quad \text{and} \quad (A+\delta A)(x+\delta x) = (b+\delta b)$$

then

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A)}{1-\kappa(A)\cdot\frac{\|\delta A\|}{\|A\|}}\left(\frac{\|\delta A\|}{\|A\|}+\frac{\|\delta b\|}{\|b\|}\right)$$

where

$\kappa(A)$ is the condition # $\|A^{-1}\|\,\|A\|$.

This gives us one interpretation of the condition number. Here's another:

Theorem. Let A be nonsingular. Then

$$\min\left\{\frac{\|\delta A\|_2}{\|A\|_2} : A+\delta A \text{ singular}\right\} = \frac{1}{\kappa(A)}.$$

Proof. We claim

$$\min\{\|\delta A\|_2 : A + \delta A \text{ singular}\} = \frac{1}{\|A^{-1}\|_2}.$$

Suppose $\|\delta A\|_2 < \dfrac{1}{\|A^{-1}\|_2}$. Then

$$1 > \|\delta A\|_2 \, \|A^{-1}\|_2 \geq \| \overset{A^{-1}\cdot\delta A}{\cancel{\delta A \cdot A^{-1}}} \|_2,$$

so we know (by our previous lemma)

$I + A^{-1}\delta A$ is ~~invertible~~ nonsingular,

so

$$A(I + A^{-1}\delta A) = A + \delta A \text{ is nonsingular.}$$

This means that the minimum is at least $1/\|A^{-1}\|_2$.

Now by definition

$$\|A^{-1}\|_2 = \max_{x \neq 0} \frac{\|A^{-1}x\|_2}{\|x\|_2}$$

We compute

$$\|\delta A\|_2 = \max_{z \neq 0} \frac{\| x y^T z \|_2}{\|A^{-1}\|_2 \|z\|_2}$$

$$= \max_{z \neq 0} \frac{\| x \|_2}{\|A^{-1}\|_2} \frac{| y^T z |}{\| z \|_2}$$

where $y^T z$ is the dot product. But this max is clearly 1 by Cauchy-Schwartz when z is a scalar moltiple of y.

So

$$\|\delta A\|_2 = \frac{\|x\|_2}{\|A^{-1}\|_2} = \frac{1}{\|A^{-1}\|_2},$$ since x is unit.

Now we claim $A + \delta A$ is singular:

$$(A + \delta A) y = Ay - \frac{x y^T y}{\|A^{-1}\|_2}$$

$$= A\left(\frac{A^{-1} x}{\|A^{-1}\|_2}\right) - \frac{x}{\|A^{-1}\|_2} = 0.$$

This completes the proof. $\square$

So there's a vector x so that $\|x\|_2 = 1$ and $\|A^{-1}x\|_2 = \|A^{-1}\|_2$. Since A^{-1} is nonsingular, this norm is > 0.

So let

$$y = \frac{A^{-1}x}{\|A^{-1}x\|_2} = \frac{A^{-1}x}{\|A^{-1}\|_2}$$

This is clearly a unit vector. Now x and y are $n \times 1$ column vectors. We are used to writing

$$x \cdot y = x^T y = (1 \times n) \cdot (n \times 1) = (1 \times 1)$$

be we can also construct the <u>$n \times n$ matrix</u>

$$x\, y^T = (n \times 1) \cdot (1 \times n) = (n \times n)$$

Now let

$$\delta A = \frac{-x y^T}{\|A^{-1}\|_2}.$$

Another approach: Suppose $\hat{x}$ is arbitrary. Then if

$$Ax = b,$$

we have

$$\delta x \equiv \hat{x} - x = \hat{x} - A^{-1}b.$$

We can bound this by setting

$r = A\hat{x} - b$ to be the residual of $\hat{x}$.

Then

$$\delta x = A^{-1}r = A^{-1}(A\hat{x} - b) = \hat{x} - x.$$

so

$$\|\delta x\| = \|A^{-1}r\| \leq \|A^{-1}\| \cdot \|r\|.$$

We like this because r is easy to compute.

Theorem. Let $r = A\hat{x} - b$. $\exists$ a δA with $\|\delta A\| = \frac{\|r\|}{\|\hat{x}\|}$ and $(A+\delta A)\hat{x} = b$. No δA of smaller norm with $(A+\delta A)\hat{x} = b$ exists.

So $\|\delta A\|$ is the smallest possible backward error in A.