Convergence of Jacobi, GS and SOR(ω).

When do these methods converge? How fast?

Definition. A is <u>strictly</u> <u>row</u> <u>diagonally dominant</u> if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$.

Theorem. If A is strictly row diagonally dominant, then both Jacobi and GS converge. GS converges "faster" in the sense that $\| R_{GS} \|_\infty \leq \| R_J \|_\infty < 1$.

Proof. We have $R_J = L + U = (D^{-1}(\tilde{L} + \tilde{U}))$ and $R_{GS} = (I - L)^{-1} U$. We want to show

$$\| R_{GS} \| = \| \, |R_{GS}| \vec{1} \, \|_\infty \leq \| \, |R_J| \vec{1} \, \|_\infty = \| R_J \|_\infty .$$

where $\vec{1}$ is the vector of all ones. (Thus $A\vec{1}$ is the vector of row sums, etc.)

In fact, we show the stronger componentwise inequality (for all i)

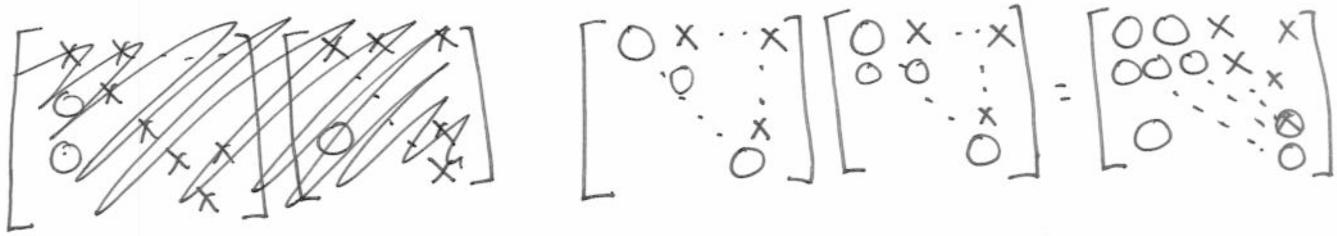
$$(|R_{GS}|\,\vec{1})_i \le (|R_J|\,\vec{1})_i$$

Now we have

$$|(I-L)^{-1}U|\cdot\vec{1} \le \underbrace{|(I-L)^{-1}|\,|U|}\cdot\vec{1}$$

triangle inequality, since these are sums of absolute values and lhs is abs. vals. of sums.

Now L is strictly lower triangular, and so



we can compute that the subdiagonals empty out as we take powers of L. In fact, $L^n = 0$. ~~So~~ Now we can expand

$$(I-L)^{-1} = \sum_{i=0}^{\infty} L^i$$

as usual and truncate the series at L^{n-1}.

We learn that

$$(I-L)^{-1} = \sum_{i=0}^{n-1} L^i$$

so our previous

$$|(I-L)^{-1}|\,|U|\,\vec{1} = \left|\sum_{i=0}^{n-1} L^i\right|\,|U|\,\vec{1}$$

$$\leq \sum_{i=0}^{n-1} |L|^i\,|U|\,\vec{1}$$

again, triangle inequality.

$$= (1-|L|)^{-1}\cdot|U|\cdot\vec{1}$$

since $|L|^n = 0$ as well.

So this means it is ok to prove (componentwise)

$$(I-|L|)^{-1}|U|\,\vec{1} \leq (|L|+|U|)\,\vec{1}$$

Now $(I-|L|)^{-1}$ is a matrix with nonnegative entries, ~~as~~ since it's $\sum |L|^i$. So if we can prove

$$|U|\,\vec{1} \leq (I-|L|)(|L|+|U|)\,\vec{1}$$

we could obtain the inequality by multiplying both sides by $(I-|L|)^{-1}$.
So we want to show

$$|U|\,\vec{1} \leq (I-|L|)(|L|+|U|)\,\vec{1}$$

or

$$0 \leq \left[(I-|L|)(|L|+|U|) - |U|\right]\vec{1}$$

$$\leq \left[\,|L| + \cancel{|U|} - |L|^2 - |L||U| - \cancel{|U|}\,\right]\vec{1}$$

$$\leq \left[\,|L| - |L|^2 - |L||U|\,\right]\vec{1}$$

$$\leq |L|\left(I - |L| - |U|\right)\vec{1}$$

But again, this will be true if

$$0 \leq (I - |L| - |U|)\,\vec{1}$$

~~or~~

Now $I \cdot \vec{1} = \vec{1}$, while $(|L|+|U|)\cdot\vec{1}$ = the vector of row sums of $|L|+|U|$. But such a sum is equal to $\sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|$, which is < 1

by the assumption that A is strictly row diagonally dominant.

Now in fact $|L| + |U| = R_J$ so if A is S.R.D.D. then $\|R_J\|_\infty$ = max row sum of $R_J < 1$. Thus we have (unwinding this whole mess) that

$$\|R_{GS}\|_\infty \leq \|R_J\|_\infty$$

as claimed. Now we have shown that in this operator norm

$$\|R_{GS}\|_\infty \leq \|R_J\|_\infty < 1,$$

so GS and Jacobi both converge. $\square$

In most cases, we can settle for

Definition. A is <u>weakly</u> diagonally dominant if $|a_{jj}| \geq \sum_{j \neq k} |a_{jk}|$ and strict inequality occurs at least once.

We need a technical hypothesis: think about the variables x_i as nodes in a graph. x_i and x_j are ~~connecting~~ <u>connected</u> if $a_{ij} \neq 0$, this allows information to flow from x_i to x_j as we iterate. ~~If the~~ If the graph is disconnected, it could split into a bunch of independent problems, Some of which might not have <u>any</u> strict inequalities.

Theorem. (see Demmel, Thm 6.3) If A is irreducible (i.e. the graph is connected) and weakly diagonally dominat, GS and Jacobi converge and $\rho(R_{GS}) < \rho(R_J) < 1$.

What about SOR(ω)? Recall

$$R_{SOR(\omega)} = (I-\omega L)^{-1}\left((1-\omega)I + \omega U\right).$$

Theorem. $\rho(R_{SOR(\omega)}) \geq |\omega - 1|$. Thus $0 < \omega < 2$ for convergence.

Proof. Consider the characteristic polynomial of $R_{SOR(\omega)}$:

$$\begin{aligned}
\varphi(\lambda) &= \det(\lambda I - R_{SOR(\omega)}) \\
&= \det\left(\lambda I - (I-\omega L)^{-1}((1-\omega)I + \omega U)\right) \\
&= \det\Big(\underbrace{(I-\omega L)}_{\text{has det } 1}\left(\lambda I - (I-\omega L)^{-1}((1-\omega)I + \omega U)\right)\Big) \\
&= \det\left((\lambda + \omega - 1)I - \omega\lambda L - \omega U\right)
\end{aligned}$$

So the constant term in $\varphi(\lambda)$, is given by

$$\begin{aligned}
\varphi(0) &= \det((\omega-1)I - \omega U) \\
&= \det((\omega - 1)I) = (\omega-1)^n
\end{aligned}$$

Now the constant term in any polynomial is the product of the roots, so this means that if $\lambda_1, \dots, \lambda_n$ are the eigenvals of $R_{SOR(\omega)}$, then

$$\prod_{i=1}^{n} \lambda_i = (\omega - 1)^n$$

So one of the λ_i must ~~be~~ have $|\lambda_i| \geq |\omega - 1|$, as desired. $\square$

It turns out to be the case that when A is symmetric positive definite,

Theorem. A s.p.d. $\Rightarrow \rho(R_{SOR(\omega)}) < 1$ for all $\omega \in (0,2)$ so $SOR(\omega)$ converges. (For $\omega = 1$, note this shows GS also converges.)

This depends on an amazing theorem.

Spectral Mapping Theorem.

Suppose p is a rational function. If A is a square matrix with eigenvalues $\{\lambda_i\}$ then $P(A)$ has eigenvalues $\{P(\lambda_i)\}$.

Proof (of SOR(ω) theorem) The splitting for SOR(ω) is

check

$$A = M - K$$
$$= \frac{1}{\omega}(D - \omega \tilde{L}) - \frac{1}{\omega}((1-\omega)D + \omega \tilde{U}$$

So let

$$Q = A^{-1}(2M - A).$$

We claim $\mathrm{Re}\,\lambda_i(Q) > 0$ for all eigenvalues λ_i of Q. Suppose $Qx = \lambda x$. Then

$$(2M - A)x = \lambda A x$$

or (as a scalar equation)

$$x^*(2M - A)x = \lambda\, x^* A x$$

Now the conjugate transpose of this equation is

$$(x^*(2M - A)x)^* = \lambda^* (x^* A x)^*$$

or

$$x^*(2M^* - A)x = \lambda^* (x^* A x)$$

since A is symmetric (and real). So if we add this to the equation above,

$$x^*(2M + 2M^* - 2A)x = (\lambda^* + \lambda)(x^* A x)$$

or

$$x^*(M + M^* - A)x = (\mathrm{Re}\,\lambda)\, x^* A x.$$

We can now solve for

$$\operatorname{Re}\lambda = \frac{x^*(M+M^*-A)x}{x^* A x}$$

Now

$$M+M^* = \omega^{-1}(D-\omega\tilde{L}) + \omega^{-1}(D-\omega\tilde{L}^*)$$

$$= \omega^{-1}(2D-\omega\tilde{L}-\omega\tilde{U})$$

b/c A symmetric

$$= \omega^{-1}((2-\omega)D + \omega D-\omega\tilde{L}-\omega\tilde{U})$$

$$= \omega^{-1}((2-\omega)D + \omega A)$$

$$= \frac{(2-\omega)}{\omega}D + A$$

So

$$M+M^*-A = \left(\frac{2}{\omega}-1\right)D$$

We know $\omega<2$, so this is a <u>positive</u> multiple of D.

Thus $(M^{*}+M^{*}-A)$ is positive definite, as is A, so

$$\mathrm{Re}\,\lambda = \frac{x^{*}(M+M^{*}-A)x}{x^{*}Ax} > 0.$$

Now we claim $(Q-I)(Q+I)^{-1} = R_{SOR(\omega)}$. This can just be checked:

<— do so —>

So we have for each eigenvalue λ of Q, the corresponding eigenvalue of R has ~~absolute value~~ complex norm

$$\left|\frac{\lambda-1}{\lambda+1}\right| = \left|\frac{(\mathrm{Re}\,\lambda-1)^2+(\mathrm{Im}\,\lambda)^2}{(\mathrm{Re}\,\lambda+1)^2+(\mathrm{Im}\,\lambda)^2}\right|^{1/2}$$

which is <1 because $\mathrm{Re}\,\lambda > 0$. This proves that $R_{SOR(\omega)}$ has spectral radius less than 1 as desired.