Error Analysis for ODE Solvers

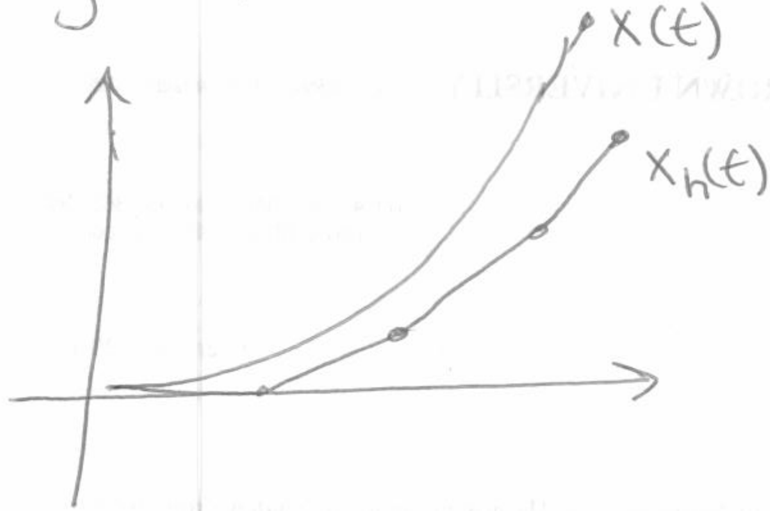
Generally speaking, it is quite hard to estimate global error for ODE solvers. If $x'(t) = f(t,x)$ and $X(t)$ is the exact solution while $x_h(t)$ is the numerical solution determined by taking steps of size h, then for Euler integration we have (Atkinson, p 346)

Theorem. If $X(t)$ has bounded second derivative on $[t_0, t_1]$ then $x_h(t)$ obeys

$$\max_{t_0 \le t \le t_1} |X(t) - x_h(t)| \le |X(t_0) - x_h(t_0)| e^{(t_1 - t_0)K} + \frac{e^{(t_1-t_0)K} - 1}{K} \cdot \|X''\|_\infty \cdot \frac{h}{2}$$

as long as $|f(t,x_1) - f(t,x_2)| \le K|x_1 - x_2|$ on $[t_0, t_1]$ (for all x_1, x_2).

Why? The idea here is that



once $x_h(t)$ diverges from $x(t)$, the error compounds, as $x_h(t+h)$ is based on $f(t, x_h(t))$ while $x(t+h)$ is based on $f(t, x(t))$. From this point of view, it's not surprising that the error grows at a rate based on

$$|f(t,x(t)) - f(t,x_h(t))| \leq K|x(t) - x_h(t)|$$

and hence is exponential in $t_1 - t_0$.

The bound

$$\|x''\|_\infty \frac{h}{2}$$

is easily recognized as the error ~~B~~ in approximating $x(t)$ by a line segment.

For the RK4 method,

$$x(t) - x_h(t) = D(t)\,h^4 + O(h^5)$$

where $D(t)$ satisfies a certain initial value problem, (which is unfortunately not nice to state). ~~However, for a fixed order~~ We have in general:

Theorem. (Atkinson, 427)
If an RK method has a step error T ~~of~~ which is $O(h^{m+1})$ then for any fixed $[t_0, t_1]$ we have $x_h(t_1) \to x(t_1)$ at rate $O(h^m)$.

Keep in mind that this result assumes $f(t,x)$ obeys a strong Lipschitz condition as above:

$$|f(t,x_1) - f(t,x_2)| < K\,|x_1 - x_2|.$$

We also note that the <u>constant</u> in this convergence is again expected to be exponential in $t_1 - t_0$, so it may easily happen that reasonable error is not obtained w/o super high precision arithmetic.

We now consider the (easier) problem of estimating and controlling the error at the step from x_n to x_{n+1}.

In general, we write all the RK methods in the form

$$x_{n+1} = x_n + hF(t_n, x_n, h; f)$$

where the original ODE is $x'(t) = f(t,x)$. Now if we let the error at the nth step be

$$T_n(x) = [x(t_{n+1}) - x(t_n)] - hF(t_n, x(t_n), h; f)$$

then for the mth order method

$$T_n(x) = \varphi(t_n)h^{m+1} + O(h^{m+2})$$

where $\varphi(t)$ ~~does~~ is a function of $x(t)$ and $f(t,x)$. So now observe that T_n for the order $m+1$ method is $\ll T_n$ for the order m method.

Thus at any step

$$\left|(\text{RKM result}) - (\text{RK}(M+1)\text{ result})\right| \approx \left|\text{RKM error}\right|$$

This is the basis for the very popular Runge-Kutta-Fehlberg methods, such as RKF45. This is given by

$$K_1 = hf(t, x)$$

$$K_2 = hf\left(t + \frac{1}{4}h, x + \frac{1}{4}K_1\right)$$

$$K_3 = hf\left(t + \frac{3}{8}h, x + \frac{3}{32}K_1 + \frac{9}{32}K_2\right)$$

$$K_4 = hf\left(t + \frac{12}{13}h, x + \frac{1932}{2197}K_1 - \frac{7200}{2197}K_2 - \frac{7296}{2197}K_3\right)$$

$$K_5 = hf\left(t + h, x + \frac{439}{216}K_1 - 8K_2 + \frac{3680}{513}K_3 - \frac{845}{4104}K_4\right).$$

$$K_6 = hf\left(t + \frac{1}{2}h, x - \frac{8}{27}K_1 + 2K_2 - \frac{3544}{2565}K_3 + \frac{1859}{4104}K_4 - \frac{11}{40}K_5\right)$$

We can then compute

the 4th order formula

$$x(t+h) \approx x(t) + \frac{25}{216}K_1 + \frac{1408}{2565}K_3 + \frac{2197}{4104}K_4 - \frac{1}{5}K_5$$

and the 5th order formula

$$x(t+h) \approx x(t) + \frac{16}{135}K_1 + \frac{6656}{12825}K_3 + \frac{28561}{56430}K_4 - \frac{9}{50}K_5 + \frac{2}{55}K_6$$

subtracting them gives our error estimate ε.

~~It is then true~~

We then proceed as follows:

Given $\varepsilon_{min}, \varepsilon_{max}, h_{min}, h_{max}, x(t), t, h$.

Compute $x(t+h)$ and ε as above.

If $\varepsilon < \varepsilon_{min}$, replace h by $2h$ (or h_{max})

If $\varepsilon > \varepsilon_{max}$, replace h by $h/2$ (or h_{min})

In either case, retry the step with new h.

Otherwise, accept the step with this h.

We now try this out!