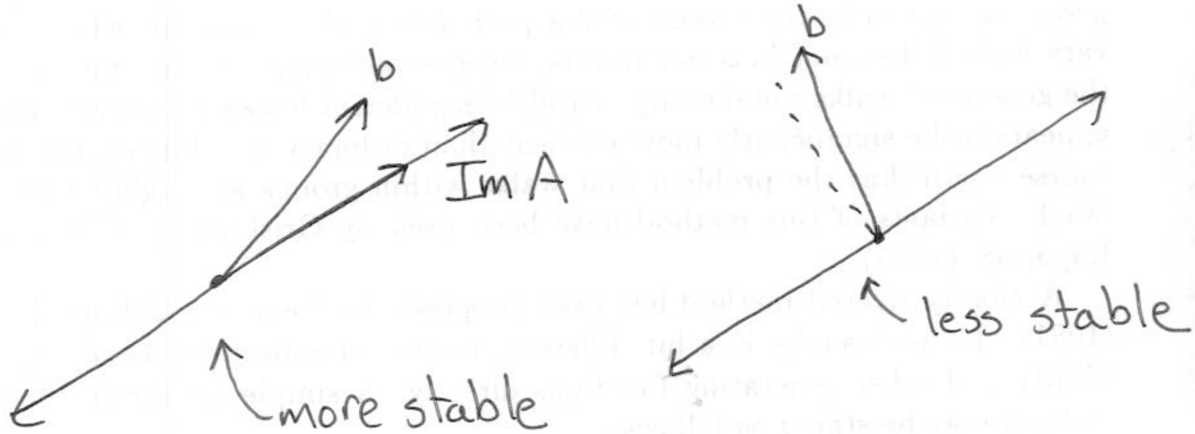
Condition Number for Least Squares.

It is an interesting feature of least squares that the difficulty of solving a given problem may depend on the rhs b and not just on A.



We can still define a "condition number"

Definition. If A is $m\times n$, $m>n$, we define

$$K_2(A) := \sigma_1(A)/\sigma_n(A),$$

the ratio of largest and smallest condition numbers.

Theorem. Suppose A is $m \times n$ with $m \geq n$ and full rank. If x minimizes $\|Ax-b\|_2$, then let $r = Ax-b$ be the residual. Let $\tilde{x}$ minimize $\|(A+\delta A)\tilde{x} - (b+\delta b)\|_2$. If

$$\epsilon \equiv \max\left(\frac{\|\delta A\|_2}{\|A\|_2}, \frac{\|\delta b\|_2}{\|b\|_2}\right) < \frac{1}{\kappa_2(A)} \text{ then}$$

$$\frac{\|\tilde{x}-x\|_2}{\|x\|_2} \leq \epsilon \cdot \left\{\frac{2 \cdot \kappa_2(A)}{\cos\theta} + \tan\theta \; \kappa_2^2(A)\right\} + O(\epsilon^2)$$

where $\sin\theta = \frac{\|r\|_2}{\|b\|_2}$ or θ is the angle between b and Ax.

Proof. We know

$$\tilde{x} = \left((A+\delta A)^T (A+\delta A)\right)^{-1} (A+\delta A)(b+\delta b).$$

Expand this in δA and δb and consider only linear terms. $\square$

It turns out that there's a nicer form here for the error in the residual.

$$\frac{\|\tilde{r} - r\|_2}{\|r\|_2} \leq \left(1 + 2\epsilon \kappa_2(A)\right).$$