The Romberg Algorithm.

We have now given a recursive formula for the trapezoid rules with 2^n intervals:

$$R(n,0) = \frac{1}{2} R(n-1,0) + h \sum_{k=1}^{2^{n-1}} f[a + (2k-1)h]$$

where $h = \frac{b-a}{2^n}$.

~~We observe that we have~~

$$\cancel{\int_a^b f(x)\,dx - R(n-1,0) = -\frac{1}{12}(b-a)h^2 f''(\xi)}$$

~~but here ξ depends on h, a, b, n.~~

We have also proved (as a corollary of the Euler-Maclaurin formula) that

$$\int_a^b f(x)\,dx = R(n-1,0) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \dots$$

where the a_i depend on f but not h. (Recall they are combinations of constants, Bernoulli numbers and odd derivatives of f evaluated at b and a.)

Idea: Let's use Richardson extrapolation to combine $R(n,0)$, $R(n-1,0)$ to kill the leading error term, and repeat to make it better.

Recall the general form for Richardson extrapolation:

$$R(n,m) = \frac{4^m}{4^m - 1} R(n,m-1) - \frac{1}{4^m-1} R(n-1,m-1)$$

which lets us compute

$$\begin{tikzcd}
R(1,0) & & \\
R(2,0) & R(2,1) \arrow[l] \arrow[ul] & \\
R(3,0) & R(3,1) \arrow[l] \arrow[ul] & R(3,2) \arrow[l] \arrow[ul]
\end{tikzcd}$$

and the theorem

$$R(n,m) = \int_a^b f(x)\,dx + \sum_{k=m+1}^{\infty} A(k,m) \left(\frac{h}{2^n}\right)^{2k}$$

So the estimate $R(n,m)$ has error $O(h^{2m+2})$.

Of course, this should be tried out on our favorite examples!

<Romberg_integration.nb>