Romberg Integration.

We ~~now~~ Know that

$$\mathrm{Trap}(f,n) - \int_a^b f(x)\,dx = \frac{1}{12} f''(\xi)\frac{(b-a)^3}{n^2}$$

Let's rewrite this in terms of the width of the interval $h = \frac{b-a}{n}$.

$$\mathrm{Trap}(f,n) - \int_a^b f(x)\,dx = \frac{1}{12} f''(\xi)(b-a)h^2$$

There's a tantalizing hint here - what if the trapezoid was an estimator whose error was a sum of even powers of h? Then we could use Richardson extrapolation to improve the results.

Theorem. For any K,

$$\mathrm{Trap}(f,n) - \int_a^b f(x)\,dx = \sum_{i=1}^{K} c_i h^{2i} + o(h^{2K})$$

where $h = \frac{b-a}{n}$.

The proof is in the graduate notes. We even derive a formula for the c_i.

Definition. The Bernoulli numbers B_i are given by $B_i = \sum_{k=0}^{i} \sum_{v=0}^{k} (-1)^v \binom{k}{v} \frac{v^i}{k+1}$ or by $\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}$.

Then in the theorem above, we may take

$$c_i = -\frac{B_{2i}}{(2i)!}\left(f^{(2i-1)}(b) - f^{(2i-1)}(a)\right)$$

where $f^{(k)}$ is the k-th derivative of f.

where $f^{(2i-1)}(x)$ is the $(2i-1)$st derivative of f, as usual.

This has the amazing consequence ~~that~~

Corollary. Suppose that

$$f^{(2i-1)}(b) = f^{(2i-1)}(a)$$

for $i = 1, \ldots, m$. Then

$$\text{Trap}(f,n) - \int_a^b f(x)\,dx = c_{2m+2} h^{2m+2} + O(h^{2m+4})$$

In particular, if $f(x)$ is periodic and has period $\frac{b-a}{k}$ (for some integer k), then the trapezoid rule <u>converges</u> <u>faster</u> <u>than</u> <u>any</u> <u>power</u> <u>of</u> <u>h</u>.

With that in mind, let's revisit our puzzling example (and some new ones!)

(error rate of trapezoid rule)

Now we're going to turn our attention to exploiting our theorem which reveals Trap(f,n) as an estimator which can be used with Richardson extrapolation. We let

$$R(n,0) = \mathrm{Trap}(f, 2^n)$$

and ~~e~~recall the recursive formula

$$R(n,m) = \frac{4^m}{4^m - 1} R(n, m-1) - \frac{1}{4^m - 1} R(n-1, m-1)$$

which lets us compute (for example)

$$
\begin{tikzcd}
R(1,0) & & \\
& R(2,1) \arrow[ul] \arrow[dl] & \\
R(2,0) & & R(3,2) \arrow[ul] \arrow[dl] \\
& R(3,1) \arrow[ul] \arrow[dl] & \\
R(3,0) & &
\end{tikzcd}
$$

By our previous theorem on Richardson extrapolation (Notes 12, p.7)

$$R(n,m) = \int_a^b f(x)\,dx + \sum_{k=m+1}^{\infty} A(k,m)\left(\frac{b-a}{2^n}\right)^{2k}$$

So the $R(n,m)$ estimate has error $O\left(\left(\frac{b-a}{2^n}\right)^{2m+2}\right)$. This procedure is almost an algorithm called <u>Romberg integration</u>.

Observe that

$$R(n,0) = \text{Trap}(f, 2^n)$$
$$= \frac{b-a}{2^{n+1}}\left(f(a) + \sum_{i=1}^{2^n-1} 2f\left(a + i\left(\frac{b-a}{2^n}\right)\right) + f(b)\right)$$

But if we've already computed

$$R(n-1,0) = \text{Trap}(f, 2^{n-1})$$
$$= \frac{b-a}{2^n}\left(f(a) + \sum_{i=1}^{2^{n-1}-1} 2f\left(a + i\left(\frac{b-a}{2^{n-1}}\right)\right) + f(b)\right)$$

we've done half the work already!

Working out the algebra,

$$R(n,0) = \frac{b-a}{2^{n+1}}\left(f(a) + \sum_{j=1}^{2^{n-1}-1} 2f\left(a + \overbrace{2j}^{\text{even } i}\left(\frac{b-a}{2^n}\right)\right) + f(b)\right)$$
$$+ \frac{b-a}{2^{n+1}}\left(\sum_{j=1}^{2^{n-1}} 2f\left(a + \overbrace{(2j-1)}^{\text{odd } i}\left(\frac{b-a}{2^n}\right)\right)\right)$$
$$= \frac{1}{2}R(n-1,0) + \frac{b-a}{2^{n+1}}\sum_{j=1}^{2^{n-1}} 2f\left(a + (2j-1)\left(\frac{b-a}{2^n}\right)\right).$$

This means that computing the Romberg integration estimate

$$R(n,n-1)$$

takes time $O(2^n)$, which is the <u>same</u> as computing Trap$(f,2^n)$. But the Romberg error is $O\left(\left(\frac{b-a}{2^n}\right)^{2n}\right)$ while the Trapezoid error is $O\left(\left(\frac{b-a}{2^n}\right)^{2}\right)$.

So let's try this out

(romberg integration data)

We can write down an error formula for Romberg integration like our formula for Trap.

Theorem.

$$R(n,m) - \int_a^b f(x)\,dx = r_m h^{2m+2} (b-a) f^{(2m+2)}(\xi)$$

where $\xi \in (a,b)$, $h = \frac{b-a}{2^n}$, and

$$r_m = \frac{2^{(m+1)m} |B_{2m+2}|}{(2m+2)!}$$

Of course, the case we really care about is

$$R(n,n-1) - \int_a^b f(x)\,dx = \frac{2^{n(n-1)} |B_{2n}|}{(2n)!} h^{2n} (b-a) f^{(2n)}(\xi)$$

(here B_i is the Bernoulli number). This tells us that the error is bounded in terms of the $2n$-th derivative of f.