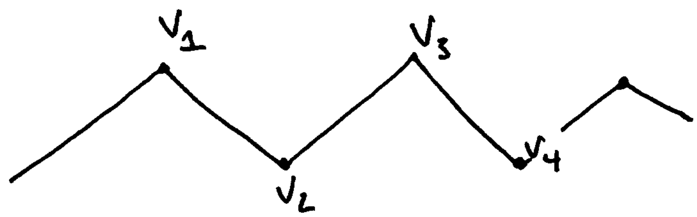
Application: Geometric Derivatives

Suppose we have a space curve $\vec{r}(s)$

which we approximate by a polygonal line



How should we write down the tangent vector of the polygonal line at a vertex? How should we write the curvature or torsion of the polyline?

We start with a brief review of the differential geometry of curves.

Suppose $\vec{r}(s)$ is parametrized so that $|\vec{r}\,'(s)| \equiv 1$. Then we say $\vec{r}$ is parametrized by arclength, since the length of the portion of r between s_0 and s_1 is

$$\int_{s_0}^{s_1} |\vec{r}\,'(s)|\, ds = s_1 - s_0.$$

In this case, we can write

$\vec{r}\,'(s) = T(s)$ ← the unit tangent vector.

Now the rate of change of $T(s)$ measures how fast the curve is turning, so we let

$$|T'(s)| = \kappa \leftarrow \text{the } \underline{\text{curvature}} \text{ of } r$$

and define the <u>normal</u> vector of $r(s)$ by

$$T'(s) = \kappa N(s).$$

Now the derivative of $N(s)$ must be perpendicular to $N(s)$ since

$$\langle N(s), N(s) \rangle \equiv 1$$

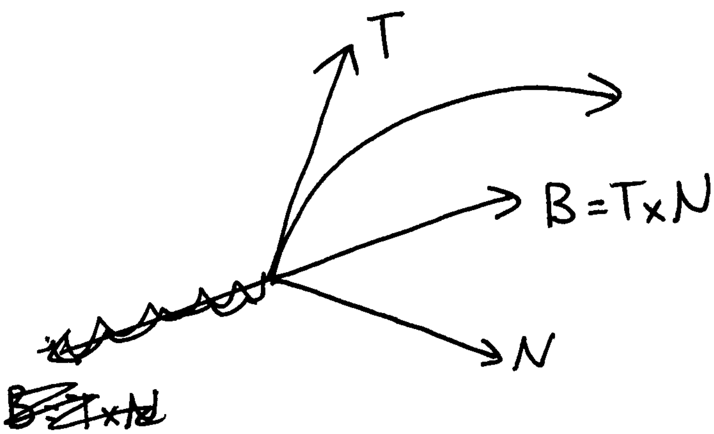
so

$$\frac{d}{ds}\langle N(s), N(s) \rangle = 2\langle N'(s), N(s) \rangle = 0.$$

Thus

$$N'(s) = a(s)\, T(s) + b(s)\, B(s),$$

where $B(s) = T(s) \times N(s)$ is the third vector in ~~our~~ the ~~s~~ orthogonal system of coordinates



Now we can compute

$$0 = \frac{d}{ds} \langle T(s), N(s) \rangle$$

$$= \langle T'(s), N(s) \rangle + \langle T(s), N'(s) \rangle$$

$$= \langle K(s) N(s), N(s) \rangle + a(s)$$

$$= K(s) + a(s).$$

So $a(s) = -K(s)$.

On the other hand, $b(s)$ measures the rate at which $N(s)$ is spinning around $T(s)$ so we call it <u>torsion</u> $\tau(s)$.

Thus, to recap

$$
\begin{aligned}
T'(s) &= \qquad\quad \kappa N(s) \\
N'(s) &= -\kappa T(s) \qquad\quad + \tau B(s)
\end{aligned}
$$

and we can compute

$$
\begin{aligned}
0 = \frac{d}{ds}\langle B(s), N(s)\rangle &= \langle B'(s), N(s)\rangle + \langle B(s), -\kappa T(s) + \tau B(s)\rangle \\
&= \langle B'(s), N(s)\rangle + \tau.
\end{aligned}
$$

while

$$
\begin{aligned}
0 = \frac{d}{ds}\langle B(s), T(s)\rangle &= \langle B'(s), T(s)\rangle + \langle B(s), T'(s)\rangle \\
&= \langle B'(s), T(s)\rangle + 0.
\end{aligned}
$$

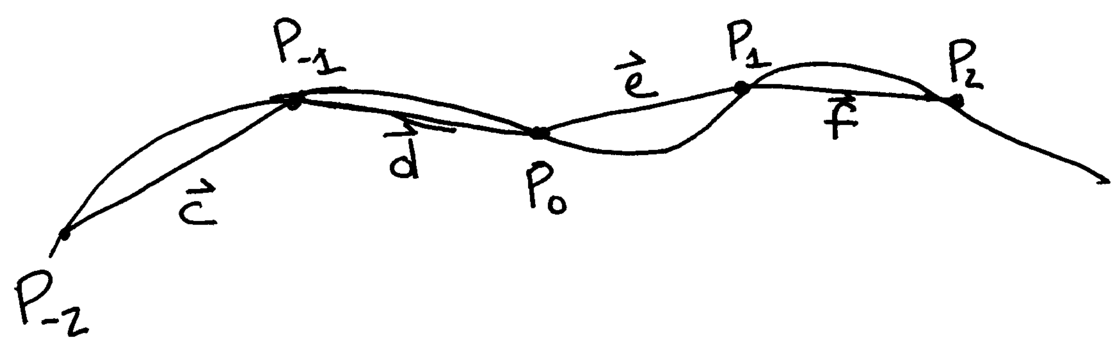
So our last equation becomes

$$B'(s) = -\tau N(s).$$

~~It is very important to have estimates~~

As you can see, curvature and torsion determine everything about how the frame $T(s), N(s), B(s)$ evolves ~~over~~ ~~as~~ as s changes.

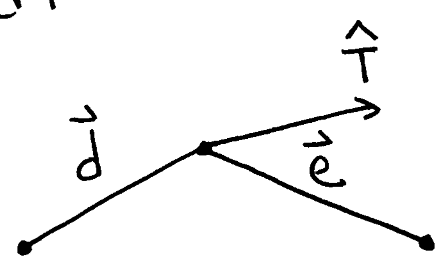
How do we define $T(s), N(s), B(s), \kappa(s)$ and $\tau(s)$ for polylines? Assume



and $c = |\vec{c}|$, $d = |\vec{d}|$ and so forth.

We now report some recent (2005) results of Langer, Belyaev, Seidel on this problem:

First, given



How should we combine $\vec{d}$ and $\vec{e}$ to get an approximation to T (called $\hat{T}$)? It's natural to assume we should take

$$\frac{\vec{d}+\vec{e}}{2} \quad \text{(weight by length)}$$

$$\frac{\vec{d}}{d}+\frac{\vec{e}}{e} \quad \text{(weight equally)}$$

and normalize them.

In fact:

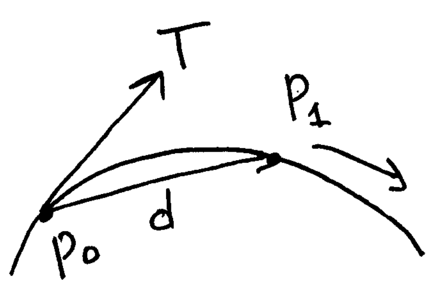
Theorem [LBS, 2005]

The only linear combination of $\vec{d}$ and $\vec{e}$ which yields a second order approximation to T is given by

$$\hat{T} = \frac{de}{d+e}\left(\frac{\vec{d}}{d^2} + \frac{\vec{e}}{e^2}\right).$$

This is the tangent to the circle through P_{-1}, P_0, P_1.

The proof is a clever and involved Taylor series calculation. The intuition, though, is simple: as P_1 moves away from P_0 along $r(s)$, the secant d gets farther from T and should have less influence on $\hat{T}$.



How should we compute KN? The first idea is to try to approximate the difference between succesive tangents.

Theorem [LBS 2005]

$$\hat{KN} := \frac{2}{d+e}\left(\frac{\vec{e}}{e} - \frac{\vec{d}}{d}\right)$$

Is a linear approximation to KN and a quadratic approximation if $d=e$. Further if φ is the turning angle from $d+e$,

$$\hat{K} := \left|\frac{2}{d+e}\left(\frac{\vec{e}}{e} - \frac{\vec{d}}{d}\right)\right|$$

or

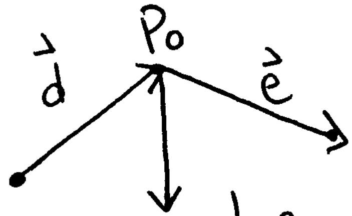
$$\hat{K} := \frac{2\varphi}{d+e}$$

have the same asymptotic error as $\hat{KN}$.

What about torsion? Well, we know

$$B'(s) = \tau N(s).$$

And given three points we can estimate B by the normal of the plane through those points.



$$b_0 = \frac{d \times e}{|d \times e|}$$

Continuing, we have

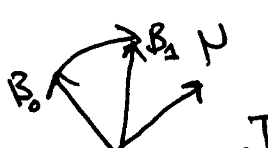
$$b_1 = \frac{e \times f}{|e \times f|}, \qquad b_{-1} = \frac{c \times d}{|c \times d|}.$$

We ~~than~~ can then define change vectors

$$\hat{\eta}_\epsilon = \langle b_1 \times b_0, \hat{T} \rangle$$

$$\hat{\eta}_\delta = \langle b_{-1} \times b_0, \hat{T} \rangle$$

These measure the approximate norm of B' since we know if B_0 is heading in the N direction as it swings to B_1, then $B_0 \times B_1$ is ~~norm~~ parallel to T.



We then have

Theorem [LBS 2005]
A linear approximation to torsion is given by

$$\hat{\tau}_d := \frac{3\hat{\eta}_d}{c+d+e}$$

and

$$\hat{\tau}_e := \frac{3\eta_e}{d+e+f}.$$

We can get a better estimate for torsion in terms of these quantities by (essentially) Richardson extrapolation, leading to a 5-point quadratic approximation for τ.

~~Step 1: Combine $\hat{\tau}_d$ and $\hat{\tau}_e$ to eliminate leading error terms:~~

$$\cancel{\hat{\tau} := \frac{1}{c+d+e+f}\left((f+2e+d)\hat{\tau}_d + (c+2d-e)\hat{\tau}_e\right).}$$