Introduction to error analysis.

The numbers stored in a computer represent approximations of real numbers. We will need to understand the effect of repeated approximation during a calculation.

Definition. A real interval is a closed, connected set of real numbers.

Examples.

$[-1,2]$, $[5, 3\pi]$, $[-2,+\infty)$

Definition. (Hickey/Ju/Emden)
If X and Y are real intervals,

$$X+Y = \{x+y \mid x\in X \text{ and } y\in Y\}$$
$$X-Y = \{x-y \mid x\in X \text{ and } y\in Y\}$$
$$X\times Y = \{xy \mid x\in X \text{ and } y\in Y\}$$
$$X/Y = \{z \mid \exists x\in X, y\in Y \text{ with } y\neq 0 \text{ so } z = x/y\}.$$

We note that $X+Y$, $X-Y$ and $X\times Y$ are real intervals, but X/Y may not be, if Y contains 0.
Example.

$$\frac{[1,1]}{[-1,\infty]} = \underbrace{(-\infty,1] \cup [0,\infty)}_{\text{neither closed nor connected}}$$

We can work out some examples:

$$[2,2] \times [\pi,\pi] =$$

$$[0,0] \times (-\infty,\infty) =$$

$$[0,1] / [0,1] =$$

Idea: We can describe the <u>accuracy</u> of $[a,b]$ as an approximation to some (unknown) number inside $[a,b]$ ~~as~~ either

absolutely $\iff$ error $= |b-a|$

relatively $\iff$ error $= \dfrac{|b-a|}{\left|\left(\frac{b+a}{2}\right)\right|}$

To get a sense of why we might prefer one or the other, we consider some examples.

Examples.

$$\underbrace{[10,11]}_{\text{absolute error }1} \times \underbrace{[100,101]}_{\text{absolute error }1} = \underbrace{[1{,}000,1111]}_{\text{absolute error }111}$$

$$\underbrace{[10,11]}_{\text{absolute error }1} + \underbrace{[100,101]}_{\text{absolute error }1} = \underbrace{[110,112]}_{\text{absolute error }2}$$

$$\underbrace{[10,11]}_{\text{absolute error }1} - \underbrace{[100,101]}_{\text{absolute error }1} = \underbrace{[-91,-89]}_{\text{absolute error }2}$$

$$\underbrace{[10,11]}_{\text{absolute error }1} / \underbrace{[100,101]}_{\text{absolute error }1} = \underbrace{\left[\frac{10}{101},\frac{11}{100}\right]}_{\text{absolute error }\frac{111}{10100}\approx 0.01}$$

This is not very satisfactory! (Although addition and subtraction are pretty ok.)

We now consider these from the point of view of relative error

$$[10,11] \times [100,101] = [1000, 1110]$$

$re = \frac{10}{105} \approx 0.1$ $\quad$ $re = \frac{10}{1005} \approx 0.01$ $\quad$ $re = \frac{110}{1055} \approx 0.11$

$$[10,11] + [100,101] = [110, 112]$$

$re \approx 0.1$ $\quad$ $re \approx 0.01$ $\quad$ $re = \frac{2}{111} \approx 0.02 \; (\lesssim 0.1)$

$$[10,11] - [100,101] = [-91,-89]$$

$re \approx 0.1$ $\quad$ $re \approx 0.01$ $\quad$ $re = \frac{2}{90} \approx 0.022 \; (\lesssim 0.1)$

$$[10,11] / [100,101] = \left[\frac{10}{101}, \frac{11}{100}\right]$$

$re \approx 0.1$ $\quad$ $re \approx 0.01$ $\quad$ $re = \frac{222}{2111} \approx 0.105$

Also not very easy! Though multiplication ~~obey~~ and division ≈ preserve the worst relative error in input numbers,

- This is also true for addition and subtraction. (though we could do better).

Principle 1. Use relative error, not absolute error.

We now consider other weird features of this new world.

Ex. $X^2 + X \neq (X + \frac{1}{2})^2 - \frac{1}{4}$

To convince yourself of this, work out both sides for $X = [-1, 2]$

$\cancel{[-1,1] \times [-1,1] + [-1,1]}$

$\cancel{= [0,1] + [-1,1]}$

$\cancel{= [-1,2]}$

~~rhs:~~ $\cancel{([-1,1] + [\tfrac{1}{2}, \tfrac{1}{2}]}$

lhs: $[-1,2]\times[-1,2]+[-1,2]$

$=[-2,4]+[-1,2]$

$=[-3,8]$, abs error ~ 11, rel error ~ 4.4

rhs: $[-\tfrac{1}{2},\tfrac{5}{2}]\times[-\tfrac{1}{2},\tfrac{5}{2}]-[\tfrac{1}{4},\tfrac{1}{4}]$

$=[-\tfrac{5}{4},\tfrac{25}{4}]-[\tfrac{1}{4},\tfrac{1}{4}]$

$=[-\tfrac{3}{2},6]$, abs error $\sim 7\tfrac{1}{2}$, rel error $\sim 3\tfrac{1}{3}$

Ex. $X/X \neq [1,1]$

This is obvious, but a good demonstration.

$[3,4]/[3,4]=[\tfrac{3}{4},\tfrac{4}{3}]$.

Principle 2: Calculations which would be equivalent for real numbers may <u>not</u> be equivalent for approximate numbers.

We note that interval arithmetic has been implemented carefully in several computing systems. If done right, it ~~can~~ gives <u>proved</u> bounds on the output of a numerical calculation!

(See paper by Hickey et. al on course page for more.)