Numerical Analysis (4500/6500)

The Key idea of this class is simple:

1) If you want to solve an actual, real-world problem, then somebody, somewhere is going to have use a computer to calculate a number.

2) Because that number was calculated in a finite approximation to the real numbers, it will be wrong.

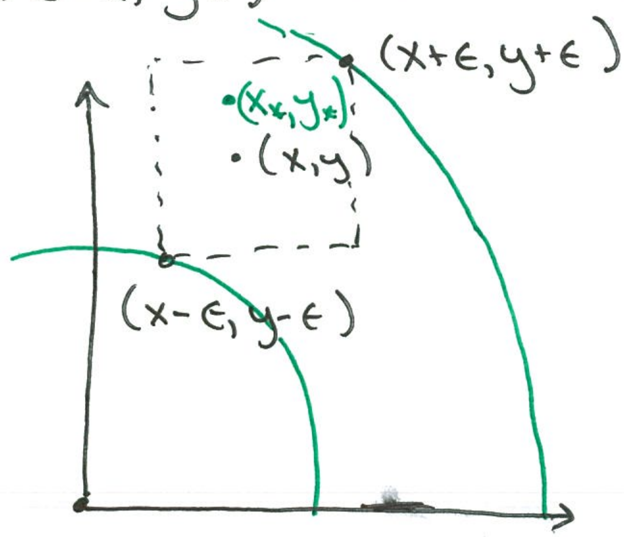
3) Determining exactly how wrong the number is, and improving the situation, is the subject of this class.

A "simple" example.

A point $(x_*, y_*) = p$ has distance

$$d(x_*, y_*) = \sqrt{x_*^2 + y_*^2}$$

from $(0,0)$. Given approximate values (x, y) within ϵ of (x_*, y_*) find bounds on $d(x_*, y_*)$ in terms of x, y, and ϵ.



Idea. Taylor's Theorem!

$$f(x+\epsilon) \approx f(x) + f'(x)\,\epsilon$$

$$f(x+\epsilon_1, y+\epsilon_2) \approx f(x,y) + \nabla f(x,y) \cdot (\epsilon_1, \epsilon_2)$$

Doing the computation

$$\frac{\partial d}{\partial x} = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial d}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

so

$$\nabla d(x,y) = \frac{1}{\sqrt{x^2+y^2}}(x,y)$$

From the picture,

$$d(x-\epsilon, y-\epsilon) \leq d(x_*, y_*) \leq d(x+\epsilon, y+\epsilon)$$

~~so~~ and by Taylor's Theorem,

$$d(x+\epsilon, y+\epsilon) \approx d(x,y) + \nabla d(x,y) \cdot (\epsilon, \epsilon)$$

$$\approx \sqrt{x^2+y^2} + \frac{x\epsilon + y\epsilon}{\sqrt{x^2+y^2}}$$

$$\approx \sqrt{x^2+y^2} + \left(\frac{x+y}{\sqrt{x^2+y^2}}\right)\epsilon$$

Now we need to bound $\frac{x+y}{\sqrt{x^2+y^2}}$.

Trick: write $(x,y) = (r\cos\theta, r\sin\theta)$.

Then

$$\frac{x+y}{\sqrt{x^2+y^2}} = \frac{r\cos\theta + r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

$$= \cos\theta + \sin\theta$$

This is maximized when $\theta = \pi/4$ and has max value $\sqrt{2}$.

So (approximately)

$$d(x_*, y_*) < \sqrt{x^2+y^2} + \sqrt{2}\epsilon$$

and similarly

$$\sqrt{x^2+y^2} - \sqrt{2}\epsilon < d(x_*, y_*).$$

Check. Let $\epsilon = 0.01$ and $(x,y) = (3,4)$.

We know

$$\sqrt{(x-\epsilon)^2+(y-\epsilon)^2} \leq d(x_*, y_*) \leq \sqrt{(x+\epsilon)^2+(y+\epsilon)^2}$$

$$4.986 \leq d(x_*, y_*) \leq 5.014$$

$\sqrt{2.99^2+3.99^2}$ $\sqrt{3.01^2+5.01^2}$

We compare to

$$\sqrt{3^2+4^2} - \sqrt{2}\times(0.01) = 5 - \sqrt{2}\times 0.01$$

$$= 4.9586$$

and

$$\sqrt{3^2+4^2} + \sqrt{2}\times(0.01) = 5 + \sqrt{2}\times(0.01)$$

$$= 5.01414$$

Pretty good!

We see from this example that tracking the propagation of error through a calculation can be challenging! (even a very simple)

Definition. Given α and β, if α is correct and β is an approximation to α,

$|\alpha - \beta|$ is the <u>absolute</u> error of β.

$\dfrac{|\alpha-\beta|}{|\alpha|}$ is the <u>relative</u> error of β.

We usually use relative error, especially when dealing with small α. We can see the difference when we round answers in a computation to a certain number of digits.

Example.

3.1415926 is rounded to 3.1416

abs error $= 7.4 \times 10^{-6}$

rel error $= 2.35549 \times 10^{-6}$ or 0.00023% error

0.0003151 is rounded to 0.0003

abs error $= 1.51 \times 10^{-5}$

rel error $= 4.79213 \times 10^{-2}$ or 4% error

huge!

Now we define

significant digits = digits beginning with leftmost nonzero digit and ending with rightmost correct digit (counting trailing zeros if accurate).

Definition. We say β is <u>correct to n significant digits</u> if the first n digits of β are correct.

Example.

3.1416 is correct (as an approx to 3.1415926) to 4 significant digits

0.0003 is correct (as an approx to 0.0003151) to only 1 significant digit.

Definition. We say β is <u>correct to n decimal places</u> if n digits to the right of the decimal point are correct.

Example.

3.1416 is correct to 3 decimal places
0.0003 is correct to 4 decimal places

Moral: Don't trust calculations accurate to n decimal places unless you know that everything in the calculation is larger than 10^{-n}.

We now look at a Mathematica demonstration of the effect of precision on computation of the solution of a 2x2 system of linear equations

(precision_and_linear_equations.nb)