Errors in Polynomial Interpolation

Suppose we want to approximate a (nice) function $f(x)$ with a polynomial $p(x)$ on ~~any~~ a fixed interval $[a,b]$.

We choose nodes $x_0, \dots, x_n$ in $[a,b]$, and compute. Clearly $f(x)=p(x)$ at $x_0, \dots, x_n$.

Let

$$E(n) = \max_{x \in [a,b]} |f(x) - p(x)|.$$

It is natural to expect $E(n) \to 0$ as $n \to \infty$. In fact, for evenly spaced nodes, it is often the case that $E(n) \to \infty$ as $n \to \infty$!

Case study: $f(x) = \frac{1}{1+x^2}$ on $[-1,1]$.

<chebyshev-nodes.nb>

We can give a few theorems about polynomial interpolation to shed some light on the matter.

Theorem. If p is the degree (at most) n polynomial interpolating f at $n+1$ distinct nodes $x_0, \dots, x_n$ in $[a,b]$ and $f^{(n+1)}$ is continuous, then for each $x \in [a,b)$ there is some $\xi \in (a,b)$ so

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i).$$

Proof. If x is a node, we're done. If not, let us fix x and set

$$\omega(t) = \prod_{i=0}^{n} (t - x_i)$$

$$c = \frac{f(x) - p(x)}{\omega(x)}$$ ← this is a constant

$$\varphi(t) = f(t) - p(t) - c\,\omega(t).$$

(Since x is not a node, $\omega(x) \neq 0$, so c exists.)

Now $\varphi(t) = 0$ at all the nodes $x_0, \dots, x_n$ and x.

We recall Rolle's theorem:

Theorem. Between any two roots of f there is a root of f'.

The interval $[a,b]$ now contains

$n+2$ roots of φ
$n+1$ roots of φ'
n roots of φ''
$\vdots$
1 root of $\varphi^{(n+1)}$

At this point (call it ξ), we have

$$0 = \varphi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - p^{(n+1)}(\xi) + c\,\omega^{(n+1)}(\xi).$$

Now p is a polynomial of degree n, so $p^{(n+1)}(x) \equiv 0$. Now $\omega(t)$ is a polynomial of degree $n+1$ with leading term t^{n+1}, so $\omega^{(n+1)}(t) \equiv (n+1)!$ So

$$0 = f^{(n+1)}(\xi) + c\,(n+1)!$$

Substituting in the definition of c... $\square$

Of course, this is not very helpful unless we can estimate $\prod (x-x_i)$.

Lemma. If $x_i = a + ih$, $i \in 0, \dots, n$, $h = \frac{b-a}{n}$, then for any $x \in [a,b]$,

$$\prod_{i=0}^{n} |x - x_i| \le \frac{1}{4} h^{n+1} n!$$

Proof. If $x \in [x_j, x_{j+1}]$ we can show (homework) that

$$|x - x_j||x - x_{j+1}| \le \frac{h^2}{4}.$$

so

$$\prod_{i=0}^{n} |x - x_i| \le \left(\prod_{i=0}^{j-1} (x - x_i) \right) \frac{h^2}{4} \left(\prod_{i=j+2}^{n} (x_i - x) \right)$$

Now for $x_i \in [x_0, \dots, x_{j-1}]$, we have

$x_{j+1} > x > x_j > x_i$, so $x - x_i < x_{j+1} - x_i$.

Similarly for $x_i \in [x_{j+2}, x_n]$ we have

$$x_i > x_{j+1} > x > x_j, \text{ so } x_i - x < x_i - x_j.$$

So we have

$$\leq \left(\prod_{i=0}^{j-1} x_{j+1} - x_i \right) \frac{h^2}{4} \left(\prod_{i=j+2}^{n} x_i - x_j \right)$$

But $x_{j+1} - x_i = (j+1-i)h$ and $x_i - x_j = (i-j)h$. So we really have

$$\leq h^j \cdot \frac{h^2}{4} \cdot h^{n-(j+2)+1} \prod_{i=0}^{j-1} (j-i)+1 \prod_{i=j+2}^{n} i-j$$

$$\leq \frac{h^{\cancel{2}+\cancel{j}+n-\cancel{j}-\cancel{2}+1}}{4} (j+1)!\,(n-j)!$$

$$\leq \frac{1}{4} h^{n+1} n!$$

where we use $(j+1)!\,(n-j)! \leq n!$ for $j \in 0, \ldots, n$. For j "in the middle" this is a gross overestimate but for $j = 0$ we can do no better. $\square$

We can now combine these to get

Theorem. If $f^{(n+1)}$ is continuous on $[a,b]$ and bounded by M on $[a,b]$, then

$$|f(x)-p(x)| \le \frac{1}{4(n+1)} M h^{n+1}.$$

We can see from this that the key question is the growth of M as a function on n.

Interpolation Errors and Divided Differences.

Last time we saw that in general for a polynomial we had

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i)$$

Our mathematica examples show us that this error bound can blow up, along with the actual error.

<gridpoints_and_interpolation.nb>

Theorem. The Chebyshev nodes $\hat{x}_i$ minimize

$$\max_{x \in [a,b]} \left| \prod (x - \hat{x}_i) \right|$$ over all choices of x_i.

In fact, for Chebyshev nodes we have

$$|f(x) - P_n(x)| \leq \frac{1}{2^n (n+1)!} M,$$ where M bounds $f^{(n+1)}(x)$

(note that this last bound only works on $[-1,1]$).

Now we show

Theorem. If p interpolates f on $x_0, \dots, x_n$ then for any x which is not a node,

$$f(x) - p(x) = f[x_0, x_1, \dots, x_n, x] \prod_{i=0}^{n} (x - x_i).$$

Proof. We know that if q interpolates f ~~and~~ at $x_0, \dots, x_n$ and t then

$$q(x) = p(x) + f[x_0, \dots, x_n, t] \prod_{i=0}^{n} (x - x_i)$$

but at $t = x$, $q = f$, so we have

$$f(x) - p(x) = f[x_0, \dots, x_n, x] \prod_{i=0}^{n} (x - x_i). \quad \square$$

Now this is weird and interesting because it suggests a connection between divided differences and derivatives:

Theorem. If $f^{(n)}$ is continuous on $[a,b]$, for any (distinct) $x_0, \ldots, x_n$ in $[a,b]$ there is some ξ in (a,b) so

$$f[x_0, \ldots, x_n] = \frac{1}{n!} f^{(n+1)}(\xi).$$

Proof. Just combine these formulae for $f(x) - p(x)$. $\square$

Corollary. If f is a polynomial of degree n, then $f[x_0, \ldots, x_i] = 0$ for $i > n$.

Example. Are the data points

x	1	-2	0	3	-1	7
y	-2	-56	-2	4	-16	376

formed by sampling a cubic?

(compute divided diffs.)