Gauss Quadrature

(First page missing)

If we integrate

$$\int_a^b p(x)\,dx = \sum f(x_i) \int_a^b l_i(x)\,dx$$

so these integrals must be the Newton-Cotes weights

What if we choose other x_i?

Theorem. (Gauss Quadrature)

Let q be any (nontrivial) polynomial of degree $n+1$ so that

$$\int_a^b x^k q(x)\,dx = 0$$

for all $k \in 0, \ldots, n$ (integers k). Let $x_0, \ldots, x_n$ be the roots of q.

then

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

where $A_i = \int_a^b \ell_i(x)\,dx$ will be exact for all polynomials of degree $\leq 2n+1$.

This is shocking! But we aren't agreeing everywhere with these polynomials, just coming up with the same integral.

Proof. Let f be any polynomial of degree $\leq 2n+1$. We can write

$$f = pq + r,$$

where $p(x)$ and $r(x)$ have degree at most n.

Now we know

$$\int_a^b f(x)\,dx = \int_a^b p(x)\,q(x) + r(x)\,dx$$

but $p(x)$ has degree $\leq n$, so

$$\int_a^b p(x)\,q(x)\,dx = 0.$$

Further, since degree $r \leq n$, we have

$$\int_a^b r(x)\,dx = \sum_i r(x_i) \int_a^b l_i(x)\,dx$$

$$= \sum_i r(x_i)\,A_i$$

(that is, r is integrated exactly by Newton-Cotes on the nodes x_i). But the x_i are <u>roots</u> of q, so

$$r(x_i) = p(x_i)\,q(x_i) + r(x_i) = f(x_i)$$

so $\sum_i r(x_i) A_i = \sum_i f(x_i) A_i$, as desired. $\square$

Now we need to find these magic polynomials $q(x)$. Notice that these polynomials can be obtained using the idea that polynomials of degree n are a vector space of dimension $n+1$ with inner product (on (a,b))

$$\langle p(x), q(x) \rangle = \int_a^b p(x)q(x)\,dx.$$

Observation. $\exists$ a set of orthogonal polynomials $p_n(x)$ so that

- $p_i(1) = 1$
- p_n has degree n

We can find these polynomials by Gram-Schmidt orthogonalization.